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Complexity bounds for cylindrical cell decompositions of sub-Pfaffian sets

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I dedicate this thesis to my loving parents.

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Summary

The fact that the first order theory of the real ordered field admits quantifier elimination is an easy consequence of Tarski-Seidenberg principle, which asserts that the projection of a semialgebraic set is semialgebraic. This important property of the class of semialgebraic sets (first proved by Tarski around 1940) plays a key role in many aspects and applications of real algebraic geometry. Over the last few decades, and as a result of intensive research efforts, some very efficient quantifier elimination methods appeared. One natural question which then arises is whether it is possible to extend the general framework by replacing polynomials with real analytic functions and still maintain quantifier elimination results. It turns out that this is no longer attainable, even if we only allow exponential functions. A weaker statement though, is sometimes true in this case, the possibility to eliminate one sort of quantifiers, either $\forall$ or $\exists$ (Gabrielov's theorem of the complement).

In this thesis we present a method which decomposes the closed unit cube $I^n \subset \mathbb{R}^n$ into a disjoint union of cylindrical cells, compatible with a given semianalytic subset $S \subset I^n$, in such a way that if S is described by members of any family of restricted analytic functions closed under addition, multiplication and taking partial derivatives, then each cell of the decomposition is a subanalytic set described by functions from the same family. In the important particular case when the analytic functions involved in the definition of S come from a certain broad finitely defined class (namely, the class of Pfaffian functions) we are able to actually construct an algorithm for producing such a cylindrical cell decomposition, provided we are given an oracle for deciding emptiness of semi-Pfaffian sets. This implies the possibility of effective elimination of one sort of quantifiers from a first-order formula involving restricted Pfaffian functions. The complexity of the algorithm as well as the bounds on parameters of the output are doubly exponential in $O(n^2)$ and are the best up-to-date. An improved estimate, doubly exponential in $O(n)$, on the number of cells in cylindrical decompositions of semi-Pfaffian sets is also established. Note that the corresponding best estimate for the semialgebraic case (which follows from Collins' CAD method) is essentially of the same order.

Glossary

I = the closed unit interval $[0,1]$;

$\mathbb{N}$ = the set of natural numbers;

$\mathbb{Q}$ = the field of rational numbers;

$\mathbb{R}_{alg}$ = the field of algebraic numbers;

$\mathbb{R}$ = the field of real numbers;

$\mathbb{R}_k$ = a nonstandard elementary extension of the field of real numbers $(k > 0)$;

$\mathbb{C}$ = the field of complex numbers;

$\mathcal{L}_r$ = the language of rings $(+, \cdot, -, 0, 1)$;

$\mathcal{L}_{or}$ = the language of ordered rings $(+, \cdot, -, 0, 1, <)$;

$\mathcal{C}$ = the $\mathcal{L}_r$-structure $(\mathbb{C}, +, \cdot, -, 0, 1)$;

$\mathcal{R}$ = the $\mathcal{L}_{or}$-structure $(\mathbb{R}, +, \cdot, -, 0, 1, <)$;

$\mathcal{R}^{(k)}$ = the $\mathcal{L}_{or}$-structure $(\mathbb{R}_k, +, \cdot, -, 0, 1, <)$, $k \geq 0$;

$\mathcal{L}_{exp}$ = the language of ordered rings with a symbol for exponentiation;

$\mathcal{R}_{exp}$ = the $\mathcal{L}_{exp}$-structure $(\mathbb{R}, +, \cdot, -, 0, 1, <, exp)$;

An = the family of restricted analytic functions;

$\mathcal{L}_{An}$ = the language of ordered rings with symbols for each function in An;

$\widetilde{\mathcal{L}}_{An}^{(k)} = \mathcal{L}_{An} \cup \{$constant symbol for each element in $\mathbb{R}_k\}$;

$\mathcal{R}_{An}^{(k)}$ = the $\mathcal{L}_{An}$-structure $(\mathbb{R}_k, +, \cdot, -, 0, 1, <, An)$;

$\widetilde{\mathcal{R}}_{An}^{(k)}$ = the $\widetilde{\mathcal{L}}_{An}^{(k)}$-structure $(\mathbb{R}_k, +, \cdot, -, \{c\}_{c \in \mathbb{R}_k}, <, An)$;

$\mathcal{F}$ = a subalgebra of restricted analytic functions closed under partial derivation;

$\mathcal{L}_{\mathcal{F}}$ = the language of ordered rings with symbols for each function in $\mathcal{F}$;

$\widetilde{\mathcal{L}}_{\mathcal{F}}^{(k)} = \mathcal{L}_{\mathcal{F}} \cup \{$constant symbol for each element in $\mathbb{R}_k\}$;

$\mathcal{R}_{\mathcal{F}}^{(k)}$ = the $\mathcal{L}_{\mathcal{F}}^{(k)}$-structure $(\mathbb{R}_k, +, \cdot, -, 0, 1, <, \mathcal{F})$;

$\widetilde{\mathcal{R}}_{\mathcal{F}}^{(k)}$ = the $\widetilde{\mathcal{L}}_{\mathcal{F}}$-structure $(\mathbb{R}_k, +, \cdot, -, \{c\}_{c \in \mathbb{R}_k}, <, \mathcal{F})$;

$\mathcal{R}_{RP}$ = the expansion of the real ordered field by restricted Pfaffian functions;

$\mathcal{R}_P$ = the expansion of the real ordered field by unrestricted Pfaffian functions;

$C\Phi(X) \equiv \forall T \exists Y \left[\Phi(Y) \wedge \left((||X - Y||^2 < T^2) \vee (T = 0)\right)\right)$ for any first-order formula Φ;

The symbol $\square$ denotes the end or absence of a proof.

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Chapter 1

Introduction

One of the main concerns of real algebraic geometry is the study of real algebraic sets, that is, subsets of points in $\mathbb{R}^n$ (for any $n \geq 0$) satisfying some system of real polynomial equations in n variables. But the class of such sets is not closed under polynomial maps. This shifts our attention to a strictly larger class of sets for which this is true, namely the class of *semialgebraic* subsets of $\mathbb{R}^n$ whose members are defined as finite Boolean combination of solutions of real polynomial equalities and inequalities in n variables. The remarkable closure property of the class of semialgebraic sets under taking images of semialgebraic maps (that is, maps with semialgebraic graph) is known as the *Tarski-Seidenberg principle* [Tar48, Sei54]. An equivalent formulation of this result is that the first-order theory of the real ordered field admits *quantifier elimination*: any well-formed logical expression constructed from polynomial equalities and inequalities, arithmetic and logical operations $+, -, \times, \wedge, \vee, \neg$ and quantifiers $\forall$ or $\exists$ on variables, is equivalent to a similar expression which does not include quantifiers. It implies for example that the closure and frontier of a semialgebraic set is semialgebraic, and indeed it plays a key role in establishing many of the basic topological and geometric properties of the class of semialgebraic sets.

It is well-known that Tarski-Seidenberg principle is constructive. Tarski himself gave an algorithm in [Tar48] for performing quantifier elimination for the reals, however its complexity is not elementary recursive. It was not until the mid 70's, that Collins [Col75] and Wüthrich [Wüt76] proposed algorithms with significantly better worst-case running time (doubly-exponential in some polynomial in the number of variables n). In recent years the problem received a lot of attention and a number of new powerful mathematical techniques emerged, leading to the design of some very efficient quantifier elimination algorithms [Ren92, BPR96].

Łojasiewicz [Łoj64, Łoj65] considered the extension of the above theory to the analytic context. *Semianalytic* sets are defined as subsets of points in $\mathbb{R}^n$ satisfying locally the same sort of formulas which determine semialgebraic sets, but with the polynomials involved, possibly being replaced by real analytic functions defined in a common open domain $U \subset \mathbb{R}^n$. *Subanalytic* sets are locally defined as projections of relatively compact semianalytic sets. He showed that locally semianalytic sets behave much like semialgebraic sets and that subanalytic subsets of $\mathbb{R}^n$ are actually semianalytic, provided that $n \leq 2$. But examples are known (Osgood [Osg16]) of subanalytic subsets of $\mathbb{R}^3$ which do not admit semianalytic representations (even in cases when only one of the basic functions appearing in their definition is an exponential function). Thus, quantifier elimination is not generally possible in first-order theories involving real analytic functions. However a theorem, due to Gabrielov [Gab68] shows that at least one sort of quantifiers (either $\forall$ or $\exists$) can be eliminated; in geometric terms this is equivalent to saying that the complement of a subanalytic set is subanalytic. It was the key result needed to allow the extension to the subanalytic setting of various finiteness results proved by Łojasiewicz for semianalytic sets (see for example [BM88]).

The complement theorem for a specific class of analytic functions follows immediately from the existence of a *decomposition* of the ambient space, compatible with any given semianalytic set described by functions from this class, into a disjoint union of *cylindrical cells*, with each one of them being definable by existential formulas involving analytic functions from the same class.

Cylindrical cells are homeomorphic images of open balls of some dimension having a simple inductive geometrical definition. A cylindrical cell decomposition of a given set induces and is induced form a unique cylindrical cell decomposition of the projection of this set into a lower dimensional space.

The technique of cylindrical cell decomposition was employed in particular, for the construction in the algebraic case of the quantifier elimination algorithms with doubly-exponential complexity bounds that appeared in [Col75, Wüt76] (more efficient modern algorithms [GV88, Gri88, HRS90, Ren92, BPR96] do not use this technique).

In the language of mathematical logic Gabrielov's theorem of the complement can be expressed as the *model completeness* of the expansion of the real ordered field by restricted analytic functions (with the term restricted replacing the compactness conditions). Combined with finiteness results on semianalytic sets, this implies the *o-minimality* of the above structure (see also [vdD86, DvdD88]).

The notion of o-minimality (or order-minimality) is a simple finiteness condition on definable sets in one variable which has surprising consequences for definable sets in several variables (see e.g., [vdD98, Mac00]). The way that these are usually established is via existence theorems of cylindrical cell decompositions in such structures. The o-minimality framework provides a natural setting for generalizing finiteness results from semialgebraic and subanalytic geometry. It is not very successful however, in providing explicit bounds for various finite characteristics of definable sets (for example the number of their definably connected components) in terms of some kind of format or size of the input.

According to a more recent result of Gabrielov [Gab96], if the functions appearing in the definition of a given subanalytic set belong to a family of analytic functions closed under addition, multiplication, and taking partial derivatives, then the functions needed to describe its complement could be chosen from the same family.

This is the case, for instance, when the input subanalytic set is defined by *Pfaffian* functions – a result first proved by Wilkie around the same time [Wil96]. These are solutions of triangular systems of first order partial differential equations with polynomial coefficients, forming a finitely defined subclass of analytic functions which includes many important members, such as the elementary functions on appropriate subsets of their domain of definition and the sparse polynomials (or fewnomials) outside coordinate functions. Pfaffian functions were introduced by Khovanskii [Kho80, Kho91], who proved that *semi-Pfaffian* sets, defined by systems of equations and inequalities between these functions, satisfy similar global finiteness properties to those of semialgebraic sets. A natural notion of format can be associated to these functions, enabling the extention of various complexity results from the polynomial case. Basic finite geometric and topological characteristics of semi-Pfaffian sets can be explicitly estimated in terms of formats of their defining functions [Kho80, Kho91, Gab95, Gab98, GV95a, Zel99].

Cylindrical cell decompositions of *sub-Pfaffian* sets (relatively proper images of semi-Pfaffian sets) were shown to exist as part of Wilkie's model-theoretic proof of the model completeness and o-minimality of the expansion of the reals by restricted Pfaffian functions [Wil96]. The complexity estimates which can be extracted from this work are apparently non-elementary.

Recently Gabrielov and Vorobjov in [GV01], modified the methods from [Gab96] to obtain an algorithm for producing cylindrical cell decompositions of sub-Pfaffian sets in $\mathbb{R}^n$ (under the assumption that an oracle for deciding emptiness of semi-Pfaffian sets is invoked). The complexity bound of this algorithm, as well as the number and formats of cells are doubly exponential in $O(n^3)$ (assuming that each oracle call has a unit cost). As a consequence, some efficient estimates on finite global characteristics of sub-Pfaffian sets were derived.

A major concern of this thesis is the development of new techniques for constructing cylindrical cell decompositions of sub-Pfaffian sets with similar or better explicit complexity bounds.

1.1 Statement of main results

The main results of this thesis are the following.

1. The introduction of a new method which uses only simple geometrical arguments for dealing with cylindrical cell decompositions of semianalytic sets.

2. An alternative proof of Gabrielov's main Theorem in [Gab96]:

   Let $\mathcal{F}$ be a collection of restricted analytic functions (that is, functions which are the restriction to the unit cube $I^n = [0,1]^n \subset \mathbb{R}^n$ of analytic functions on some open neighbourhood of I^n) closed under addition, multiplication and taking partial derivatives. We say that a subset $S \subset \mathbb{R}^n$ is $\mathcal{F}$-semianalytic if S is the solution set of a finite Boolean combination of atomic equalities and inequalities involving members of $\mathcal{F}$; a set $W \subset \mathbb{R}^m$, $m \leq n$, is called $\mathcal{F}$-subanalytic if it is the projection of a relatively compact $\mathcal{F}$-semianalytic set $S \subset \mathbb{R}^n$. By making use of

   - the o-minimality of the expansion of the real ordered field by restricted analytic functions, and
   - Gabrielov's result [Gab96, Lemma 1] that the closure and frontier (within the unit cube) of a $\mathcal{F}$-semianalytic set are $\mathcal{F}$-semianalytic,

   we are able to show that the complement within the unit cube of a $\mathcal{F}$-subanalytic set is $\mathcal{F}$-subanalytic.

   In model-theoretic terms this is equivalent to the model completeness of the expansion of the real ordered field by a subalgebra of restricted analytic functions closed under partial derivation.

   Unlike the proof in [Gab96], our methods do not require the use of any stratification results for semianalytic sets.

3. The effective construction of cylindrical cell decompositions of sub-Pfaffian sets:

   When applied to the Pfaffian setting, the methods we develop in order to establish the results stated in items 1 and 2 above, in conjuction with finitely many applications of the following

   - Gabrielov's algorithm for computing the closure and frontier of a restricted semi-Pfaffian set [Gab96, Lemma 1],
   - Khovanskii's uniform bound on the number of connected components of a semi-Pfaffian set [Kho80, Kho91], and

- an *oracle* (that is, a procedure which we do not specify how it actually works) for deciding the feasibility of any system of Pfaffian equations and inequalities,

yield a conditional (because of the oracle) algorithm for computing complements of restricted sub-Pfaffian sets with the best up-to-date worst-case complexity bounds.

More precisely, the input of the algorithm is a semi-Pfaffian set

$$S = \bigcup_{1 \leq l \leq M} \{f_l = 0, g_{l1} > 0, \ldots, g_{lJ_l} > 0\} \subset G \subset \mathbb{R}^n \tag{1.1}$$

where f_l, g_{lj} are restricted Pfaffian functions (see Definition 3.3.1) with a common Pfaffian chain, of order r and degree (α, β), defined in an open domain $G \subset \mathbb{R}^n$ which we assume includes the unit cube I^n.

Our methods necessitate the use of infinitesimal elements $0 < \varepsilon_0 \ll \ldots \ll \varepsilon_{n-1}$ belonging to a nonstandard elementary extension $\mathbb{R}_n$ of the real field $\mathbb{R}$. Nonstandard analysis, introduced by Robinson in the mid 60's, establishes the mathematical existence of infinitesimal elements and the possibility to "transfer" first-order properties between the real field and such nonstandard extensions.

The output of the algorithm is a cylindrical cell decomposition, say $\mathcal{D}$, of I^n compatible with $S^{(n)}$, the extension of S over the field $\mathbb{R}_n$.

Each cell is described by a formula of the type

$$\pi'\Big(\bigcup_{1 \leq i \leq M'} \bigcap_{1 \leq j \leq M''} \{h_{ij} *_{ij} 0\}\Big),$$

where h_{ij} are resticted Pfaffian functions (over $\mathbb{R}$) in $n' \geq n$ variables, some of which are replaced by these infinitesimal elements, $*_{ij} \in \{=, >\}$, π' is the projection function $\pi' : \mathbb{R}_n^{n'} \longrightarrow \mathbb{R}_n^n$, and M', M'' are certain integers.

Assuming that each oracle call has a unit cost, we prove that the number of cells in $\mathcal{D}$, the bounds on the components of their format, as well as the complexity of the algorithm, are less than

$$(\alpha + \beta N)^{r^{O(n)} 2^{O(n^2)}}, \tag{1.2}$$

where $N = 1 + \sum_{1 \leq l \leq M}(J_l + 1)$.

In certain situations (one of them being the case when S is semialgebraic) each oracle call can actually be replaced by a proper deciding procedure with its corresponding complexity bound being reflected in (1.2).

As a corollary we establish the existence of a cylindrical cell decomposition of the closed unit cube I^n in $\mathbb{R}^n$ compatible with S, comprising of sub-Pfaffian sets, so that the number of cells in this decomposition and the bounds for the components of the format of each cell are at most as in (1.2) above.

For any $m \leq n$, let

$$\begin{aligned} \rho: \qquad \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ (X_1, \dots, X_n) &\longmapsto (X_{n-m+1}, \dots, X_n), \end{aligned}$$

be the projection function with $\rho(S) = W$ and denote by $\widetilde{W} = I^m \setminus W$ the complement of W within the cube I^m. By the definition (see Section §2.4 below) of a cylindrical cell decomposition, $\mathcal{D}$ induces a cylindrical cell decomposition of the cube $I^m = \rho(I^n) \subset \mathbb{R}_n^m$ compatible with $W^{(n)}$, the extension of W over $\mathbb{R}_n$.

Using an oracle, the algorithm can then decide which cells belong to $W^{(n)}$ and which to its complement $\widetilde{W}^{(n)} = I^m \setminus W^{(n)}$. Clearly, $\widetilde{W}^{(n)}$ is a finite union of some cells in $\mathcal{D}$ and, as a result, our algorithm is able to produce an existential formula with parameters from $\mathbb{R} \cup \{\varepsilon_0, \dots, \varepsilon_{n-1}\}$ defining the set $\widetilde{W}^{(n)}$.

By introducing existentially quantified variables in the place of these infinitesimals in the formula defining $\widetilde{W}^{(n)}$, and due to the elementary equivalence of the fields $\mathbb{R}$ and $\mathbb{R}_n$, it is possible to infer the existence of an existential formula with parameters from $\mathbb{R}$ defining the set $\widetilde{W} \subset \mathbb{R}^m$, thus, proving an effective version of the complement theorem for sub-Pfaffian sets.

4. The proof of a sharper upper bound on the number of cells in cylindrical cell decompositions of sub-Pfaffian sets:

   Using only Khovanskii's result regarding the number of nondegenerate solutions of a system of Pfaffian equations [Kho80, Kho91], we are able to prove that the number of cells in the cylindrical cell decomposition $\mathcal{D}$ of the unit cube I^n compatible with the semi-Pfaffian set S defined in (1.1) is at most

   $$2^{3^{2n}r^2}(n!(\alpha + 2N\beta))^{O(3^n(r+n))}, \tag{1.3}$$

   improving the estimate obtained in item 3 above.

   Notice that if S is *semialgebraic*, then the bound on the number of cells established here is essentially the same as the best known upper bound in a cylindrical cell decomposition for the polynomial case [Col75, Wüt76].

The following is an immediate consequence of the existence of such an upper bound on the number of cells in a cylindrical decomposition of the unit cube compatible with a semi-Pfaffian set. Consider a well-formed logical expression $\Psi(X_1, \dots, X_m)$ written with a finite number of conjuctions, disjunctions, negations and universal or existential quantifiers on variables (ranging over $\mathbb{R}$), starting from atomic formulas $f_i(X_1, \dots, X_n) = 0$ or $g_{ij}(X_1, \dots, X_n) > 0$, where f_i, g_{ij} are restricted Pfaffian functions in $n \geq m$ variables with a common Pfaffian chain of the order not exceeding r and degrees (α, β). If N is the number of all atomic formulas present in Ψ then the number of connected components of the subset of $\mathbb{R}^m$ defined by Ψ does not exceed (1.3).

We note that some of these results have already appeared in [PV01, PV03].

1.2 Structure of the thesis

The rest of the thesis is organised as follows.

In Chapter 2 we present some elementary definitions and results from Model Theory. In particular, we introduce the notion of o-minimal (model theoretic) structures which provides a general framework for establishing finiteness properties of definable sets in such structures similar to those of semialgebraic and subanalytic sets.

The first part of Chapter 3 serves as a brief overview of historical developments of quantifier elimination methods for the first order theory of the reals. Due to its more general practical importance as well as its relevance to the subject of this thesis, we provide a more detailed description of Collins' algorithm for producing cylindrical cell decompositions of semialgebraic sets. Moving to the analytic case next, we discuss first, several properties of semianalytic and subanalytic sets defined over the real ordered field $\mathbb{R}$. We introduce in particular, a special class of analytic functions, namely the class of Pafffian functions, and examine some finiteness results regarding semi- and sub-Pfaffian sets. Finally, we consider one important consequence of the Compactness theorem from Model Theory: the existence of nonstandard elementary extensions of the real field $\mathbb{R}$.

In Chapter 4 we describe a new method of obtaining a cylindrical cell decomposition of the closed unit cube $I^n \subset \mathbb{R}^n$ compatible with a given real semianalytic set S. This method is based on geometrical characteristics of parametric families of analytic curves closed under projection on suitable coordinate subspaces and the possibility of identifying finite sets of points on these curves (which we call *special points*) consisting of points of self-intersection and local extrema with respect to coordinate functions. We conclude the chapter with a proof of the correctness of this description.

Chapter 5 is devoted to the actual construction of the previous description in such a way that if the functions present in the formula defining the set S belong to a collection F of restricted analytic functions, then all the cells of the described decomposition are definable by existential formulas involving real analytic functions from the algebra $A(F)$ generated by the functions from F, their partial derivatives, constants 0 and 1, and coordinate functions. This is equivalent to the model-completeness of the expansion of the real ordered field by functions from $A(F)$. In order to achieve this, we need to define some large systems of equations and inequalities involving analytic functions in many variables (including the original ones) using infinitesimals to "approximate" parametric points in (almost) every cell of this decomposition. In order to "pass to the limit" we employ a result due to Gabrielov [Gab96, Lemma 1] regarding the closure and frontier (within the unit cube) of a semianalytic set determined by analytic functions belonging to a subalgebra closed under partial derivation.

In Chapter 6 we focus our attention to the case when the semianalytic set S mentioned above, is actually semi-Pfaffian. The methods developed in Chapter 5 can be adjusted accordingly so that when applied to this case, an (conditional) algorithm for performing quantifier simplification for expressions involving restricted Pfaffian functions can be obtained. The complexity bounds follow from the complexity bounds of Gabrielov's algorithm for computing the closure of a semi-Pfaffian set [Gab98] and Khovanskii's bound on the number of connected components of a semi-Pfaffian set [Kho80, Kho91]. In the remaining part of this Chapter, we use techniques similar to those employed in our proof of the complement theorem for sub-Pfaffian sets, in order to build a system of Pfaffian equations (which involves even more infinitesimal elements) defining a finite set of points in some larger space, with the following property: a suitable multiple of its cardinality constitutes an upper bound on the number of cells in the described cylindrical cell decomposition $\mathcal{D}$ of the unit cube I^n compatible with the semi-Pfaffian set S.

In Chapter 7 we discuss one particular direction for further research work. We examine the possibility of constructing effective cylindrical cell decompositions of restricted sub-Pfaffian sets endowed with a finite CW-complex structure (that is, cylindrical cell decompositions with some extra cell frontier conditions). This would imply for example, an efficient estimate for the sum of Betti numbers of such sets in terms of their format, as well as a method for computing their fundamental group.

Appendix A deals with nonstandard elementary extensions of the real field $\mathbb{R}$. It seeks to provide a better understanding of subanalytic sets defined over such fields, as well as some of the nonstandard techniques that are employed throughout this thesis, which take advantage of the existence of infinitesimal elements in these extensions.

Appendix B discusses a slighly different (to that presented in Chapter 4) cylindrical cell decomposition of the unit cube in $\mathbb{R}^n$ compatible with a given semianalytic set $S \subset \mathbb{R}^n$, that can be described without actually using the language of infinitesimals.

Finally, in Appendix C we present a method for computing the fundamental group of a CW-complex.

Chapter 2

Basic Model Theory

In this chapter we introduce some concepts and elementary results from Model Theory. This branch of mathematical logic studies mathematical structures by examining first order sentences that are true in these structures.

Of central importance in model theory is the analysis of sets definable in mathematical structures by first order formulas. Noteworthy examples of such sets are the constructible sets in algebraic geometry and semialgebraic or subanalytic sets in real geometry, all of which were studied extensively in their own right by other non-model-theoretic means. Model theory not only provides a convenient terminology for dealing with many aspects of such sets, but it also contributes to a better understanding of these objects offering significant insights, which have led to new results.

We are particularly interested in the study of o-minimal structures. O-minimality refers to a simple condition on (parametrically) definable subsets of the underlying set of a structure and was shown to have strong consequences for arbitrary definable sets in higher dimensions. The o-minimality framework provides a natural generalization to a more general abstract setting of finiteness results in semialgebraic and subanalytic geometry.

For a detailed introduction to model theory as well as current developments and applications of this field to other mathematical areas, the reader can consult, for example, [Mar02, HPS00, Pil00]. In our exposition of o-minimality we follow closely Van den Dries' recent book [vdD98]; other good references for this subject include [Mar96, Edm99] and the articles [Mar00, vdD00, Mac00] in [HPS00] – for a general discussion regarding "tameness" conditions see [Tei97].

2.1 Languages and Structures

A (model theoretic) structure $\mathcal{M}$ is a non-empty set M equipped with a collection $\{f_j^M : j \in J\}$ of n_j-ary functions on M, a collection $\{R_i^M : i \in I\}$ of n_i-ary relations on M (that is, subsets of M^{n_i} for some $n_i > 0$), and a collection $\{c_k^M : k \in K\}$ of distinguished elements of M. We use the notation

$$\mathcal{M} = \big(M, \{c_k^M\}_{k\in K}, \{f_j^M\}_{j\in J}, \{R_i^M\}_{i\in I}\big),$$

the superscripts can be omited when no confusion arises from doing so. In general, we assume that the diagonal relation $\{(y,y) : y \in M\}$ is included in the list of distinguished relations of $\mathcal{M}$.

As an example consider the ordered field of real numbers $\mathcal{R}$ which has $\mathbb{R}$ as its underlying set, binary functions $+, \cdot, -$, relation $<$ and constants 0 and 1; we write $\mathcal{R} = (\mathbb{R}, +, \cdot, -, <, 0, 1)$.

To each structure we associate a language $\mathcal{L}$ consisting of an n_j-ary function symbol f_j for each f_j^M, an n_i-ary function symbol R_i for each R_i^M, and constant symbols c_k for each c_k^M. We write $\mathcal{L} = (\{c_k\}_{k\in K}, \{f_j\}_{j\in J}, \{R_i\}_{i\in I})$.

An $\mathcal{L}$-structure is a structure $\mathcal{M}$ where we can *interpret* all of the symbols of $\mathcal{L}$.

Let $\mathcal{L}_r = (+, \cdot, -, 0, 1)$ denote the language of rings, and $\mathcal{L}_{or} = \mathcal{L}_r \cup \{<\}$ denote the language of ordered rings. Then the field of complex numbers $\mathcal{C} = (\mathbb{C}, +, \cdot, -, 0, 1)$ is an $\mathcal{L}_r$-structure, while the ordered field of real numbers $\mathcal{R}$ is clearly an $\mathcal{L}_{or}$-structure.

If $\mathcal{L}_0 \subseteq \mathcal{L}_1$ are two languages and $\mathcal{M}_0$ and $\mathcal{M}_1$ are respectively an $\mathcal{L}_0$-structure and $\mathcal{L}_1$-structure having the same underlying set M, then we say that $\mathcal{M}_1$ is an expansion of $\mathcal{M}_0$.

Let $\mathcal{L}$ be some language and $X_1, X_2, \ldots$ a countable collection of variables.

The set of $\mathcal{L}$-*terms* is defined by induction as follows:

- Constant symbols and variables are $\mathcal{L}$-terms;
- If $t_1, \ldots, t_l$ are $\mathcal{L}$-terms and f is an l-ary function symbol in $\mathcal{L}$, then $f(t_1, \ldots, t_l)$ is an $\mathcal{L}$-term.

An atomic $\mathcal{L}$-*formula* is an expression of the form $t_1 = t_2$ or $R(t_1, \ldots, t_m)$, where R is an m-ary relation symbol in $\mathcal{L}$ and $t_1, \ldots, t_m$ are $\mathcal{L}$-terms.

The set of first order $\mathcal{L}$-*formulas* is recursively defined as follows:

- Atomic $\mathcal{L}$-formulas are $\mathcal{L}$-formulas;
- If Φ, Ψ are $\mathcal{L}$-formulas then $\Phi \vee \Psi$, $\Phi \wedge \Psi$ and $\neg\Phi$ are $\mathcal{L}$-formulas;

- If Φ is an $\mathcal{L}$-formula and X_i is a variable then $\exists X_i \Phi$ and $\forall X_i \Phi$ are $\mathcal{L}$-formulas.

We emphasize that first order $\mathcal{L}$-formulas are finite strings of symbols in which variables are allowed to range only over elements of the set M. The logical connectives $\vee$ (or), $\wedge$ (and), $\neg$ (not) correspond to the operations of union, intersection and complementation of sets. The existential quantifier corresponds to taking a projection and the universal quantifier is expressible by means of the equivalence $\forall X_i(\Phi) \longleftrightarrow \neg\exists X_i(\neg\Phi)$.

We say that a variable appearing in an $\mathcal{L}$-formula Φ is *bounded*, if it is inside the scope of a quantifier. If this is not the case, we say that the variable is *free*.

If $\mathcal{M}$ is an $\mathcal{L}$-structure, then $\mathcal{L}$-formulas can be interpreted as expressing statements about $\mathcal{M}$ or properties of some n-tuples of elements of its underlying set M, for some $n > 0$.

An $\mathcal{L}$-formula Φ with no free variables is called an *$\mathcal{L}$-sentence*; it expresses a statement and it can be either *true* or *false* in any $\mathcal{L}$-structure $\mathcal{M}$. We say that $\mathcal{M}$ is a *model* of Φ if and only if Φ is true in $\mathcal{M}$; we write $\mathcal{M} \models \Phi$.

An $\mathcal{L}$-formula with free variables expresses properties and it may or may not be satisfied by some elements of the underlying set of $\mathcal{M}$. We will often write $\Phi(X_1, \dots, X_n)$ to show that the variables $X_1, \dots, X_n$ are free in Φ. If $\beta_1, \dots, \beta_n$ are elements of M, we write $\mathcal{M} \models \Phi(\beta_1, \dots, \beta_n)$ to mean that the property expressed by Φ is true for the n-tuple $(\beta_1, \dots, \beta_n)$.

We say that $\mathcal{M}$ has *quantifier elimination* if every $\mathcal{L}$-formula $\Phi(X_1, \dots, X_n)$ is equivalent in $\mathcal{M}$ to a quantifier-free $\mathcal{L}$-formula $\Psi(X_1, \dots, X_n)$, that is,

$$\mathcal{M} \models \Phi(\beta_1, \dots, \beta_n) \Longleftrightarrow \mathcal{M} \models \Psi(\beta_1, \dots, \beta_n), \qquad \forall(\beta_1, \dots, \beta_n) \in M^n.$$

It is often the case that sets determined by quantifier-free $\mathcal{L}$-formulas in $\mathcal{M}$ have "good" geometric and topological properties (for example the semialgebraic sets when $\mathcal{M} = \mathcal{R}$). In such cases quantifier elimination is a highly desirable result since it implies that these properties also hold for all sets defined by a finite number of applications of the operations of union, intersection, projection and complementation, starting from sets determined by quantifier-free $\mathcal{L}$-formulas. It is important to point out that quantifier elimination in any structure $\mathcal{M}$ is very sensitive to the choice of language $\mathcal{L}$ of $\mathcal{M}$. For example, when the field of real numbers is viewed as an $\mathcal{L}_r$-structure, it does not have quantifier elimination; nevertheless, it turns out that adding the order relation $<$ to $\mathcal{L}_r$ is enough to ensure that as an $\mathcal{L}_{or}$-structure it has quantifier elimination. Actually it is not difficult for one to show (see for example [Mar00]) that given an $\mathcal{L}$-structure $\mathcal{M}$ is always possible to come up with a language $\mathcal{L}'$ expanding $\mathcal{L}$ so that $\mathcal{M}$ viewed as an $\mathcal{L}'$-structure has quantifier elimination. Finding such a richer language $\mathcal{L}'$, while at the same time ensuring that sets determined by quantifier-free $\mathcal{L}'$-formulas in $\mathcal{M}$ are manageable, can be a difficult and tedious task.

We say that $\mathcal{M}$ is *model complete* if every $\mathcal{L}$-formula is equivalent in $\mathcal{M}$ to an existential $\mathcal{L}$-formula. When quantifier elimination fails in a structure, model completeness is the next best result we can hope for.

A key aspect of the introduction of symbols, is the ability to compare in many ways different structures that have the same language.

If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures with underlying sets M and N respectively, an $\mathcal{L}$-*homomorphism* is a map $h : M \longrightarrow N$ such that

- $h(f^{\mathcal{M}}(\alpha)) = f^{\mathcal{N}}(h(\alpha))$, for any n-ary function f in $\mathcal{L}$ and any $\alpha \in \mathcal{M}^n$;
- $\alpha \in R^{\mathcal{M}} \Longrightarrow h(\alpha) \in R^{\mathcal{N}}$, for any n-ary relation R in $\mathcal{L}$ and any $\alpha \in \mathcal{M}^n$;
- $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for any constant c in $\mathcal{L}$.

An one-to-one $\mathcal{L}$-homomorphism $h : M \longrightarrow N$ is an $\mathcal{L}$-*embedding* if $\alpha \in R^{\mathcal{M}} \Longleftrightarrow h(\alpha) \in R^{\mathcal{N}}$, for every n-ary relation R in $\mathcal{L}$ and every $\alpha \in \mathcal{M}^n$. An $\mathcal{L}$-*isomorphism* is a bijective $\mathcal{L}$-embedding. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say either that $\mathcal{M}$ is an $\mathcal{L}$-*substructure* of $\mathcal{N}$ or $\mathcal{N}$ is an *extension* of $\mathcal{M}$, and we write $\mathcal{M} \subseteq \mathcal{N}$. This is precisely the case when the interpretations in $\mathcal{M}$ of all symbols in $\mathcal{L}$ are just the restrictions of their interpretations in $\mathcal{N}$.

Let $\mathcal{M} \subseteq \mathcal{N}$. We say either that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$ or $\mathcal{M}$ is an *elementary $\mathcal{L}$-substructure* of $\mathcal{N}$, and we write $\mathcal{M} \preceq \mathcal{N}$ if

$$\mathcal{M} \models \Phi(\alpha_1, \ldots, \alpha_n) \Longleftrightarrow \mathcal{N} \models \Phi(\alpha_1, \ldots, \alpha_n),$$

for any $\mathcal{L}$-formula $\Phi(X_1, \ldots, X_n)$ and any $(\alpha_1, \ldots, \alpha_n) \in M^n$.

To illustrate this notion consider the $\mathcal{L}_{or}$-structures $\mathcal{Q} = (\mathbb{Q}, +, \cdot, -, 0, 1, <)$, $\mathcal{R}_{alg} = (\mathbb{R}_{alg}, +, \cdot, -, 0, 1, <)$ and $\mathcal{R} = (\mathbb{R}, +, \cdot, -, 0, 1, <)$, where $\mathbb{Q}, \mathbb{R}_{alg}$ and $\mathbb{R}$ denote the ordered fields of rational, real algebraic and real numbers repectively. Clearly $\mathcal{Q} \subset \mathcal{R}_{alg} \subset \mathcal{R}$ but $\mathcal{Q} \not\preceq \mathcal{R}_{alg}$ since for example $\mathcal{R}_{alg} \models \Psi$ and $\mathcal{Q} \not\models \Psi$, where Ψ is the $\mathcal{L}_{or}$-sentence $\exists X_1(X_1^2 = 2)$, while $\mathcal{R}_{alg} \preceq \mathcal{R}$.

2.2 Definability

We say that a set $S \subset M^n$ is $\mathcal{M}$-definable if there exists a first-order $\mathcal{L}$-formula $\Phi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ and elements $\beta_1, \ldots, \beta_m \in M$ such that

$$S = \{(\alpha_1, \ldots, \alpha_n) : \mathcal{M} \models \Phi(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)\}.$$

We say that S is $\mathcal{M}$-definable with parameters from a subset $B \subseteq M$, or simply definable over B (when is clear to which structure we refer), if we can choose $\beta_1, \ldots, \beta_m \in B$. Sometimes we may just say that S is definable, meaning $\mathcal{M}$-definable with parameters from M.

For example, in the structure $(\mathbb{R}, +, \cdot, -, 0, 1)$, the natural ordering $X_1 < X_2$ is definable (with no parameters) by the formula $\exists X_3(X_3 \neq 0 \wedge X_1 + X_3^2 = X_2)$, while the set $\{X_1 : X_1 < \pi\}$ is definable over any subset $B \subseteq \mathbb{R}$ which includes the element π (clearly this set can not be defined without parameters).

A *Boolean algebra* of subsets of a set M is a non-empty collection $\mathcal{E}$ of subsets of M such that if $A, B \in \mathcal{E}$ then $A \cup B \in \mathcal{E}$ and $M \setminus A \in \mathcal{E}$. Note that $\emptyset, M \in \mathcal{E}$ and $A, B \in \mathcal{E}$ imply that $A \cap B \in \mathcal{E}$.

It is possible to obtain the following set-theoretic characterization for the $\mathcal{M}$-definable sets.

Proposition 2.2.1. *Suppose that D_n is a collection of subsets of M^n for all $n \geq 1$ and $D = (D_n : n \geq 1)$ is the smallest collection such that:*

1. *each D_n is a Boolean algebra of subsets of M^n;*
2. *for all n-ary funcion symbols f of $\mathcal{L}$, the graph of $f^{\mathcal{M}}$ is in D_{n+1};*
3. *for all n-ary relation symbols R of $\mathcal{L}$, $R^{\mathcal{M}} \in D_n$;*
4. *for all $i, j \leq n$, $\{(X_1, \ldots, X_n) \in M^n : X_i = X_j\} \in D_n$;*
5. *if $A \in D_n$, then $M \times A \in D_{n+1}$;*
6. *if $A \in D_{n+1}$ and $\pi : M^{n+1} \longrightarrow M^n$ is the projection map omiting the last coordinate, then $\pi(A) \in D_n$;*
7. *if $A \in D_{n+k}$ and $\beta \in M^k$, then $\{\alpha \in M^n : (\alpha, \beta) \in A\} \in D_n$.*

Then $A \subset \mathcal{M}^n$ is $\mathcal{M}$-definable (with parameters) if and only if $A \in D_n$.

Proof. See, e.g. [Mar02, Proposition 1.3.4]. □

Let $S \subset M^n$. A *function $f : S \longrightarrow M^k$ is $\mathcal{M}$-definable (with parameters)* if its graph $\{(a, f(a)) : a \in S\} \subset M^{n+k}$ is $\mathcal{M}$-definable (with parameters). If f is $\mathcal{M}$-definable (with parameters) then its domain $S \subset \mathcal{M}^n$ and its image $f(S) \subset M^k$ are both $\mathcal{M}$-definable (with parameters).

Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be two $\mathcal{L}$-structures with underlying sets M_0 and M_1 respectively, such that $\mathcal{M}_0 \preceq \mathcal{M}_1$ (i.e., $\mathcal{M}_1$ is an elementary extension of $\mathcal{M}_0$). According to Tarski-Vaught Test (see for example [Mar02, Proposition 2.3.5]), if $\mathcal{M}_0 \subseteq \mathcal{M}_1$ then

$\mathcal{M}_0 \preceq \mathcal{M}_1$ if and only if for any $\mathcal{L}$-formula $\Phi(X_1, Y_1, \ldots, Y_n)$ and $(\alpha_1, \ldots, \alpha_n) \in M_0^n$, if there exists $b_1 \in M_1$ such that $\mathcal{M}_1 \models \Phi(b_1, \alpha_1, \ldots, \alpha_n)$ then there exists $c_1 \in M_0$ such that $\mathcal{M}_1 \models \Phi(c_1, \alpha_1, \ldots, \alpha_n)$. This implies that for every non-empty $\mathcal{M}_1$-definable set $T^{(1)} \subseteq M_1^k$ with parameters from M_0, the set $T^{(0)} = T^{(1)} \cap M_0^k$, which we call the restriction of $T^{(1)}$ to $\mathcal{M}_0$ is non-empty and is $\mathcal{M}_0$-definable (possibly with parameters). On the other hand, if $S^{(0)} \subseteq M_0^k$ is an $\mathcal{M}_0$-definable set with parameters from M_0, determined by an $\mathcal{L}$-formula Φ, then the same formula Φ determines an $\mathcal{M}_1$-definable set $S^{(1)} \subseteq M_1^k$, which we call the extension of $S^{(0)}$ to $\mathcal{M}_1$. Of course $S^{(1)} \cap M_0^k = S^{(0)}$ but $S^{(1)}$ is not the only $\mathcal{M}_1$-definable set having this property. For example, if $\mathcal{L} = \mathcal{L}_{or}$ the language of ordered rings, $M_0 = \mathbb{R}_{alg}$ (the real algebraic numbers), $M_1 = \mathbb{R}$ and $S^{(0)} = [1,5] \subset \mathbb{R}_{alg}$, then $S^{(1)} = [1,5] \subset \mathbb{R}$, but also $[1,\pi) \cup (\pi, 5] \cap \mathbb{R}_{alg} = S^{(0)}$.

2.3 First-Order Theories

An $\mathcal{L}$-theory Σ is a set of $\mathcal{L}$-sentences. If $\mathcal{M} \models \phi$ for all sentences $\phi \in \Sigma$ then we say that $\mathcal{M}$ is model of Σ and we write $\mathcal{M} \models \Sigma$. A result of fundamental importance in Model Theory is the Compactness theorem (see for example [Mar02, Theorem 2.1.4]) which says that if every finite subset of a set Σ of $\mathcal{L}$-sentences has a model, then Σ has a model. An equivalent formulation of this result, is that whenever an $\mathcal{L}$-sentence ϕ is a formal consequence of Σ then there is a finite subset T of Σ such that ϕ is already a formal consequence of T. In particular, one can use the Compactness theorem to establish the existence of "nonstandard" models: any infinite structure has a proper elementary extension (of arbitrarily large cardinality). In the mid 1960's Robinson [Rob66] invented nonstandard analysis, solving the three hundred year old problem of Leibniz by showing that the notion of infinitesimals can indeed be treated rigorously.

The elementary or first order (complete) theory $Th(\mathcal{M})$ of an $\mathcal{L}$-structure $\mathcal{M}$ is by definition the set of first order $\mathcal{L}$-sentences true in $\mathcal{M}$. It is *axiomatizable* if there exists a set of $\mathcal{L}$-sentences Σ (which we call the set of axioms of the theory) such that $\mathcal{N} \models Th(\mathcal{M}) \iff \mathcal{N} \models \Sigma$, for any $\mathcal{L}$-structure $\mathcal{N}$.

Whether or not an $\mathcal{L}$-structure $\mathcal{M}$ [has quantifier elimination / is model complete] only depends on the theory $Th(\mathcal{M})$ of $\mathcal{M}$. We say that $Th(\mathcal{M})$ [admits quantifier elimination / is model complete] if for every $\mathcal{L}$-formula $\phi(X_1, \ldots, X_n)$ there exists [a quantifier-free / an existential] $\mathcal{L}$-formula $\psi(X_1, \ldots, X_n)$ such that

$$\mathcal{M} \models \forall X_1 \cdots \forall X_n \big(\phi(X_1, \ldots, X_n) \longleftrightarrow \psi(X_1, \ldots, X_n)\big).$$

Equivalently, $Th(\mathcal{M})$ is model complete if $\mathcal{M} \preceq \mathcal{N}$ (that is, $\mathcal{N}$ is an elementary extension of $\mathcal{M}$) whenever $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{N} \models Th(\mathcal{M})$. This amounts to showing that any finite system of equalities and inequalities involving the (interpretation of the) distinguished functions of $\mathcal{L}$, with parameters from M, has a solution in $\mathcal{M}$ only if it has a solution in $\mathcal{N}$.

Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are said to be *elementarily equivalent* and write $\mathcal{M} \equiv \mathcal{N}$, if for all $\mathcal{L}$-sentences $\mathcal{M} \models \Phi$ if and only if $\mathcal{N} \models \Phi$. Clearly, $\mathcal{M} \equiv \mathcal{N}$ if and only if $Th(\mathcal{M}) = Th(\mathcal{N})$. The elementary class of models of $Th(\mathcal{M})$ is precisely the class of $\mathcal{L}$-structures elementarily equivalent to $\mathcal{M}$. It follows that, if $\mathcal{M}$ [has quantifier elimination / is model complete] then $\mathcal{N}$ [has quantifier elimination / is model complete]. If $\mathcal{M} \preceq \mathcal{N}$ then obviously $\mathcal{M} \equiv \mathcal{N}$, although the converse may not necessarily be true. For instance, consider the structures $\mathcal{M} = (M, <)$, $\mathcal{N} = (N, <)$, with underlying sets $M = \{1, 2, \dots\}$ and $N = M \cup \{0\}$. Then $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M} \equiv \mathcal{N}$ since the two structures are isomorphic, but $\mathcal{M} \not\preceq \mathcal{N}$ since $\mathcal{M} \models \forall X_1(X_1 = 1 \vee X_1 > 1)$ and $\mathcal{N} \models \exists X_1(X_1 < 1)$ (note that 1 is not a distinguished element of M or N).

We now look at some examples.

Let $\mathcal{C} = (\mathbb{C}, +, \cdot, -, 0, 1)$ denote the field of complex numbers viewed as an $\mathcal{L}_r$-structure. The quantifier-free $\mathcal{C}$-definable sets (with parameters) are exactly the constructible sets, that is boolean combinations of zerosets of polynomials over $\mathbb{C}$. According to Chevalley's theorem for algebraically closed fields (see e.g., [Łoj91]), the projection of a constructible set is constructible; this implies that the definable sets in $\mathcal{C}$ are precisely those that are quantifier-free definable in $\mathcal{C}$ (that is, $\mathcal{C}$ admits quantifier elimination). The theory $Th(\mathcal{C})$ of $\mathcal{C}$ is axiomatized by:

- the finite axioms for fields of characteristic zero, and
- the $\mathcal{L}_r$-sentences $\forall X_1 \cdots \forall X_n \exists Y\big(Y^n + X_1 Y^{n-1} + \cdots X_n = 0\big)$ for $n \in \mathbb{N}$, expressing the statement that every (non-constant) polynomial with complex coefficients has a solution in $\mathbb{C}$.

So, models of $Th(\mathcal{C})$ are precisely the algebraically closed fields of characteristic zero.

Now consider the $\mathcal{L}_{or}$-structure $\mathcal{R} = (\mathbb{R}, +, \cdot, -, <, 0, 1)$ of the ordered field of real numbers. We say that a subset of $\mathbb{R}^n$ is *semialgebraic* if it is a finite Boolean combination of solutions of polynomial equalities $g(X_1, \dots, X_n) = 0$ and polynomial inequalities $h(X_1, \dots, X_n) > 0$. Semialgebraic sets in $\mathbb{R}^n$ are exactly the quantifier-free $\mathcal{R}$-definable sets with parameters from $\mathbb{R}$. Tarski-Seidenerg principle asserts that semialgebraic sets are closed under projection; thus any $\mathcal{R}$-definable set is actually semialgebraic. Further on in this thesis (see Section §3.1) we present a short review of quantifier elimination results (with emphasis on their complexity) for the first-order theory $Th(\mathcal{R})$ of $\mathcal{R}$, which is axiomatizable by

- $\forall X \exists Y\big(X > 0 \longrightarrow X = Y^2\big)$;

- $\forall X_1 \cdots \forall X_{2n+1} \big(Y^{2n+1} + X_1 Y^{2n} + \cdots + X_{2n+1} = 0\big)$, for $n \in \mathbb{N}$;
- the finite axioms for ordered fields.

The first axiom simply states that every positive number has a square root while the second one expresses the statement that every polynomial of odd degree has a root. The class of fields which are models for $Th(\mathcal{R})$ are called *real closed fields.*

Tarski's quantifier elimination result [Tar48] proves in particular the decidability of $Th(\mathcal{R})$. This is in clear contrast with Gödel's work which yielded the undecidability of the theory of the structure $(\mathbb{N}, +, \cdot, -, 0, 1)$. His Incompleteness Theorem implies that the definable sets in this structure are complicated and do not excibit "tame" geometrical and topological characteristics similar to those enjoyed by semialgebraic sets or, for that matter, by definable sets in o-minimal structures, which we introduce next.

2.4 O-Minimality

Tarski's quantifier elimination result for the structure $\mathcal{R} = (\mathbb{R}, +, \cdot, -, <, 0, 1)$ shows in particular, that the $\mathcal{R}$-definable subsets of the ordered field $\mathbb{R}$ of the real numbers consists of finitely many points and intervals. This property was isolated by van den Dries [vdD84] who showed that "good" behaviour of definable sets in certain expansions of $\mathcal{R}$ follow just from this. As a generalization of these ideas, Pillay and Steinhorn [PS86] introduced the notion of o-minimality.

We say that a linearly ordered set $(M, <)$ is *dense* if for all $a, b \in M$ with $a < b$ there is $c \in M$ with $a < c < b$.

Definition 2.4.1. *The expansion $\mathcal{M} = (M, <, \cdots)$ of a dense linearly ordered non-empty set $(M, <)$ without endpoints (meaning that M has no largest or smallest element) is* o-minimal *or* order-minimal, *if every $\mathcal{M}$-definable subset of M is the union of finitely many points and intervals with endpoints in $M \cup \{\pm\infty\}$.*

The terminology arises from the fact that such finite unions are the smallest Boolean algebra of subsets that can be defined using order. Even though the definition of o-minimality only refers to subsets of M, surprisingly enough, it also has strong implications for definable sets of higher dimension. In particular, many of the finiteness theorems and geometric and topological properties of semialgebraic sets pass over to definable sets in o-minimal stuctures. For results in real algebraic and semialgebraic geometry the reader is referred to the excellent texts [BR90] and [BCR98].

We equip M with the interval topology (the intervals form a basis) and each Cartesian product M^n with the corresponding product topology. The (topological) closure in $\mathbb{R}^n$ of a set $S \subset \mathbb{R}^n$ is denoted by $cl(S)$ and its interior by $int(S)$. Notice that if S is $\mathcal{M}$-definable (with parameters) then $cl(S)$ and $int(S)$ are also $\mathcal{M}$-definable (with parameters). We let $\partial S = cl(S) \setminus S$ denote the frontier of S and $bd(S) = cl(S) \setminus int(S)$ denote the boundary of S.

Throughout the thesis we let $||.||$ denote the Euclidean norm.

A set $A \subset M^k$ is called bounded if $||\alpha|| = \left(\alpha_1^2 + \cdots + \alpha_k^2\right)^{1/2} < r$ for all $\alpha \in A$ and a fixed $r \in M$. Let $S \subset M^n$ and $f : S \longrightarrow M^k$ a definable continuous map. We call f *definably proper* if for each definable set $A \subset f(S) \subset M^k$ we have: A is closed and bounded in M^k implies that $f^{-1}(A) \subset S$ is closed and bounded in M^n.

A set $S \subset M^n$ is called *$\mathcal{M}$-definably connected* if S is $\mathcal{M}$-definable and is not the union of two disjoint non-empty $\mathcal{M}$-definable open subsets of S. In the particular case when the underlying set $M = \mathbb{R}$ then if $S \subset \mathbb{R}^n$ is $\mathcal{M}$-definably connected then is also connected in the Euclidean topology in $\mathbb{R}^n$. An *$\mathcal{M}$-definably connected component* of a non-empty $\mathcal{M}$-definable set $S \subset M^n$ is by definition a maximal $\mathcal{M}$-definably connected subset of S.

2.4.1 Definition of a cell decomposition

Let $\mathcal{M}$ be any o-minimal structure. Next we draw our attention on $\mathcal{M}$-definable sets of particularly simple form which we call cylindrical cells (or sometimes simply cells) and introduce the all important notion of cylindrical cell decompositions of $\mathcal{M}$-definable sets, which is central to the subsequent developement of this thesis.

Definition 2.4.2. *([Col75, vdD98]) A* cylindrical cell *is defined as follows.*

1. *Cylindrical 0-cell in M^n is an isolated point.*

2. *Cylindrical 1-cell in M is an open interval (a, b), $a, b \in M \cup \{\pm\infty\}$.*

3. *For $n \geq 2$ and $0 \leq k < n$, a cylindrical $(k+1)$-cell in M^n is either*

   (i) a section over C, that is a graph Graph(f) of a continuous bounded $\mathcal{M}$-definable function $f : C \longrightarrow M$, where C is a cylindrical $(k+1)$-cell in M^{n-1} equipped with coordinates $X_2, \ldots, X_n$, or else

   (ii) a sector over C, that is a set of the form

$$(-\infty, f) := \left\{(X_1, \ldots, X_n) \in M^n : (X_2, \ldots, X_n) \in C,\ X_1 < f(X_2, \ldots, X_n)\right\},$$

OR

$$(f, +\infty) := \left\{(X_1, \ldots, X_n) \in M^n : (X_2, \ldots, X_n) \in C,\ X_1 > f(X_2, \ldots, X_n)\right\},$$

where C is a cylindrical k-cell in M^{n-1}, and $f : C \longrightarrow M$ a continuous bounded $\mathcal{M}$-definable function,

OR

$$(f, g) := \{(X_1, \dots, X_n) \in M^n : (X_2, \dots, X_n) \in C,$$
$$f(X_2, \dots, X_n) < X_1 < g(X_2, \dots, X_n)\},$$

where C is a cylindrical k-cell in M^{n-1}, and $f, g : C \longrightarrow M$ are continuous bounded $\mathcal{M}$-definable functions satisfying

$$f(X_2, \dots, X_n) < g(X_2, \dots, X_n)$$

for all points $(X_2, \dots X_n) \in C$.

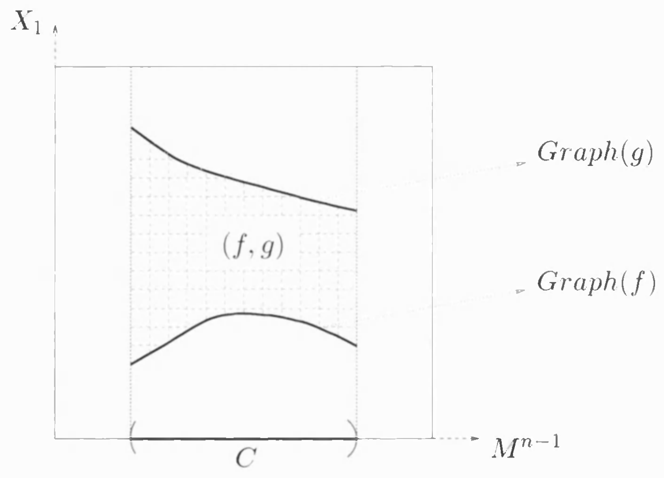


Figure 2-1: Cylindrical cells over C.

Clearly, every cylindrical k-cell is $\mathcal{M}$-definable and $\mathcal{M}$-definably homeomorphic to a product of k intervals. If in addition, the underlying set of the structure $\mathcal{M}$ is $M = \mathbb{R}$ and the semialgebraic sets are $\mathcal{M}$-definable, then every cylindrical k-cell is $\mathcal{M}$-definably homeomorphic to $\mathbb{R}^k$.

For any $\mathcal{M}$-definable set $A \subset M^n$ a *decomposition* of A is a finite collection of disjoint non empty $\mathcal{M}$-definably connected sets whose union is A.

Suppose that $f_0, f_1, \dots, f_m, f_{m+1}$ are $\mathcal{M}$-definable functions on a k-cell $C \subset M^{n-1}$, with $f_0 = -\infty, f_{m+1} = +\infty$ and $f_i(c) < f_{i+1}(c)$, $1 \leq i \leq m$, for all $c \in C$. A *stack* over C is a decomposition of the cylinder $Z(C) = C \times M = \{(c, w) : c \in C, w \in M\}$ over C into the $2m + 1$ cells given by sections $Graph(f_i)$ and sectors (f_i, f_j) over C.

Definition 2.4.3. *A* cylindrical cell decomposition, *say $\mathcal{D}$, of an $\mathcal{M}$-definable subset $A \subset M^n$ is defined as follows.*

1. *If $n = 1$, then $\mathcal{D}$ is a finite family of pair-wise disjoint cylindrical cells (i.e., isolated points and intervals) whose union is A.*

2. *If $n \geq 2$, then $\mathcal{D}$ is a finite family of pair-wise disjoint cylindrical cells in M^n whose union is A and there is a cell decomposition $\mathcal{D}'$ of $\pi(A)$ such that for each cell C of $\mathcal{D}$, the set $\pi(C)$ is a cell of $\mathcal{D}'$, where $\pi : M^n \longrightarrow M^{n-1}$ is the projection map onto the coordinate subspace of $X_2, \dots, X_n$.*

It is clear from this definition that any cylindrical cell decomposition $\mathcal{D}$ of M^n induces and is constructed from a unique cylindrical cell decomposition $\mathcal{D}'$ of the lower dimensional space M^{n-1}. We say $\mathcal{D}$ is an *extension* of $\mathcal{D}'$ if $\mathcal{D}$ induces $\mathcal{D}'$. It follows that for each cell $C' \in \mathcal{D}'$ some subset of $\mathcal{D}$ is a stack over C'.

Algorithms which produce cylindrical cell decompositions of semialgebraic sets over arbitrary real closed fields were first constructed in the mid 70's by Collins [Col75] and independently by Wüthrich [Wüt76]. Collin's work in particular has been very influential in this area; in the next Chapter we sketch the main ideas behind his Cylindrical Algebraic Decomposition method.

Definition 2.4.4. *If $A \subset M^n$, $B \subset M^n$ and $\mathcal{D}$ is a cylindrical cell decomposition of A, then $\mathcal{D}$ is* compatible *with B if for all $C \in \mathcal{D}$ either $C \subset B$ or $C \cap B = \emptyset$ (i.e. some $\mathcal{D}' \subset \mathcal{D}$ is a cylindrical cell decomposition of $B \cap A$).*

Definition 2.4.5. *If $\mathcal{D}$ and $\mathcal{D}'$ are two cylindrical cell decompositions of $A \subset M^n$, we say that $\mathcal{D}'$ is a* refinement *of $\mathcal{D}$ if for each cell $C' \in \mathcal{D}'$, the decomposition $\mathcal{D}$ is compatible with C'.*

2.4.2 Properties of definable sets in o-minimal structures

One very important consequence of the fullfilment of the o-minimality condition in some arbitrary structure $\mathcal{M}$ is the existence of cell decompositions of $\mathcal{M}$-definable sets.

Cylindrical Cell Decomposition Theorem (see e.g. [vdD98, Theorem 2.11])
Let $\mathcal{M}$ be an o-minimal structure.
(*i*) Every $\mathcal{M}$-definable set can be partitioned into finitely many disjoint (cylindrical) cells.
(*ii*) If $f : A \longrightarrow M$ is an $\mathcal{M}$-definable function, then there is a partition of S into finitely many (cylindrical) cells, such that f is continuous on each cell.

Under the assumption of "strong o-minimality", the cell decomposition theorem was proved by van den Dries [vdD84] in the case when the underlying set of $\mathcal{M}$ is the real field $\mathbb{R}$. The cell decomposition theorem for arbitrary o-minimal structures is proved by Knight, Pillay and Steinhorn in [KPS86]. In the same paper by making use of this theorem, the authors were able to obtain the following results:

1. if S be a nonempty $\mathcal{M}$-definable subset of M^n, then S has only finitely many $\mathcal{M}$-definably connected components each of which is $\mathcal{M}$-definable; moreover, they are open and closed in S and form a finite partition of S (see also [vdD98, Proposition 3.2.18]);

2. o-minimality is preserved under elementarily equivalence (that is, if $\mathcal{M}$ is o-minimal then $\mathcal{M}$ is strongly o-minimal); and

3. given an $\mathcal{M}$-definable family of $\mathcal{M}$-definable sets there is a uniform bound on the number of $\mathcal{M}$-definably connected components of the fibres in this family, that is, if S is an $\mathcal{M}$-definable subset of $M^k \times M^n$ then there is a number $\lambda_S \in \mathbb{N}$ such that for each $\alpha \in M^k$ the set $S_\alpha = \{\beta \in M^n : (\alpha, \beta) \in S\}$ has a partition into at most λ_S cells and in particular each fiber S_α has at most λ_S $\mathcal{M}$-definably connected components (see also [vdD98, Proposition 3.3.5]).

Remark 2.4.6. *Although the o-minimality framework allows one to prove the existense of such uniform bounds, it does not provide realistic bounds in terms of some kind of format of the sets in question.*

Van den Dries [vdD98, Chapter 4] uses the cell decomposition theorem to develop the definable invariant notions of o-minimal dimension and o-minimal Euler characteristic for $\mathcal{M}$-definable sets. The *dimension* of a non-empty $\mathcal{M}$-definable set S is defined by $dim(S) = max\{k : S \text{ contains a } k\text{-cell}\}$ (and $dim(\emptyset) = -\infty$), while the *Euler characteristic* of S is defined by $E(S) = \sum E(C_i)$, where $C_1, \dots, C_m$ are the members of a finite partition of S into cells and $E(C_i) = (-1)^k$ if C_i is a k-cell (and $E(\emptyset) = 0$). Both of these notions are shown to be invariant under $\mathcal{M}$-definable bijections and to possess other desirable properties.

A *stratification* of a closed set $S \subset M^n$ is a partition of S into finitely many cells $\{S_i\}$ (called strata) such that if $S_i \cap cl(S_j) \neq \emptyset$, then $S_i \subset cl(S_j)$ and $dim(S_i) < dim(S_j)$ (frontier condition). As a result, the frontier ∂S_i is a union of necessarily lower dimensional strata. It can be shown (e.g. [vdD98, Proposition 4.1.13]) that if S is an $\mathcal{M}$-definable set, then there is a stratification of S compatible with a given $\mathcal{M}$-definable subset of S.

In the case when the given o-minimal structure $\mathcal{M}$ is an expansion of an ordered abelian group, one can prove that every point in the boundary of an $\mathcal{M}$-definable set S is the limit of an $\mathcal{M}$-definable continuous curve in S. This is the so called "*Curve* *selection lemma*" ([vdD98, Corollary 6.1.5]) and it can be used, for example, to prove that the image of a closed bounded $\mathcal{M}$-definable set under a continuous $\mathcal{M}$-definable map is closed and bounded.

Under the assumption that a given o-minimal structure $\mathcal{M}$ is an expansion of an ordered field (necessarily real closed - see [vdD98, Chapter 1, (4.6)]) then one can speak of differentiability and prove a definable version of the inverse function theorem [vdD98, Chapter 7].

In this case it is also possible to show that each $\mathcal{M}$-definable bounded set S can be *triangulated* [vdD98, Theorem 8.2.9]. This means that S is definably homeomorphic to a bounded semi-linear set (that is, a bounded semialgebraic set determined by polynomials over M of degree at most 1). The triangulation theorem implies, in particular, that two definable sets are *definably equivalent* (that is, there is a definable bijection between them) if and only if they have the same dimension and the same Euler characteristic.

A *definable trivialization* of an $\mathcal{M}$-definable map $f : S \longrightarrow A$, where $A \subset M^k$ and $S \subset M^n$ are $\mathcal{M}$-definable sets, is a pair (F, h) consisting of an $\mathcal{M}$-definable set $F \subset M^l$, for some l, and an $\mathcal{M}$-definable map $h : S \longrightarrow F$, such that $(f, h) : S \longrightarrow A \times F$ is a homeomorphism. So (f, h) identifies S with the Cartesian product $A \times F$ and under this identification f corresponds to the projection map $A \times F \longrightarrow A$. Each fiber $f^{-1}(\alpha)$ of f is mapped by (f, h) homeomorphically onto $\{\alpha\} \times F$, in particular all fibers are $\mathcal{M}$-definably homeomorphic to F. We call f *definably trivial* if f has a definable trivialization. The *Trivialization theorem* for $\mathcal{M}$-definable functions (see e.g. [vdD98, Chapter 9]) states that there is a partition of the space $A = A_1 \cup \cdots \cup A_j$ into $\mathcal{M}$-definable sets A_i such that f is definably trivial over each A_i. An easy consequence of this theorem is that any given $\mathcal{M}$-definable family belongs to only finitely many *$\mathcal{M}$-definable homeomorphism types* (we say that two $\mathcal{M}$-definable sets belong to the same $\mathcal{M}$-definable homeomorphism type if there is an $\mathcal{M}$-definable homeomorphism between them).

2.4.3 Examples of o-minimal structures

What makes the theory of o-minimal structures so appealing is on one hand the validity of finiteness and "tameness" results of the kind mentioned above regarding definable sets in such structures and on the other hand the plethora of interesting examples. Here we list some of the most basic examples of o-minimal structures without getting, at this point, into any details in respect to possible methods for their justification. More will be said about most of the examples below at a later stage of the thesis.

- $\mathcal{R} = (\mathbb{R}, +, \cdot, -, 0, 1, <)$, the real orderd field (Tarski [Tar48]). The $\mathcal{R}$-definable sets (with parameters) are just the semialgebraic sets.

- $\mathcal{R}_{An} = (\mathbb{R}, +, \cdot, -, 0, 1, <, An)$, the expansion of the real ordered field by the family An of restricted analytic functions, i.e., functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ for all $n \in \mathbb{N}$ which are given on $[0,1]^n$ by a power series converging on a neighbourhood of $[0,1]^n$ and are set to 0 outside $[0,1]^n$ (Gabrielov [Gab68], see also [vdD86] and [DvdD88]). Bounded $\mathcal{R}_{An}$-definable sets are precisely the subanalytic sets (see Section §3.2 for relevant definitions), while $\mathcal{R}_{An}$-definable sets are images of bounded subanalytic sets under semialgebraic maps.

- $\mathcal{R}_{RP} = (\mathbb{R}, +, \cdot, -, 0, 1, <, F)$, the expansion of the real ordered field by a collection F of restricted Pfaffian functions which form a Pfaffian chain, see Definition 3.3.1 (Wilkie [Wil96], see also Gabrielov [Gab96]).

- $\mathcal{R}_{\mathcal{F}} = (\mathbb{R}, +, \cdot, -, 0, 1, <, \mathcal{F})$, the expansion of the real ordered field by a subalgebra $\mathcal{F}$ of restricted analytic functions closed under taking partial derivatives (Gabrielov [Gab96]). A generalization of this result appeared in [Max98].

- $\mathcal{R}_{exp} = (\mathbb{R}, +, \cdot, -, 0, 1, <, exp)$, the expansion of the real ordered field by the (unrestricted) exponential function (Wilkie [Wil96]).

- $\mathcal{R}_{An,exp} = (\mathbb{R}, +, \cdot, -, 0, 1, <, An, exp)$, the expansion of the real ordered field by the family An of restricted analytic functions, and the (unrestricted) exponential function (van den Dries, Macintyre and Marker [vdDMM94]).

- $\mathcal{R}_P = (\mathbb{R}, +, \cdot, -, 0, 1, <, F)$, the expansion of the real ordered field by a collection F of total (unrestricted) Pfaffian functions which form a Pfaffian chain, (Wilkie [Wil99]). This result is generalized in many directions and by various people (see, in particular [Spe99], [KM99]).

Establishing the o-minimality of expansions of the real ordered field $\mathbb{R}$ by even richer classes of functions is currently an active area of research.

Chapter 3

Semialgebraic and Subanalytic sets

Tarski (1948) was the first one to show that subsets of $\mathbb{R}^n$ defined by arbitrary first order formulas in the language of ordered rings can in fact be represented as finite Boolean combinations of solutions of real polynomial equalities and inequalities, and so they are semialgebraic. We begin this Chapter with a short historical overview of quantifier elimination results for the first order theory of the reals. We are particularly interested in the quantifier elimination method introduced by Collins in 1975, which is based on a cylindrical cell decomposition algorithm for semialgebraic sets. Semianalytic sets are a generalization of semialgebraic sets: they are defined locally by the same kind of formulas with analytic functions replacing the polynomials; subanalytic sets are projections of relatively compact semianalytic sets. Several properties of these sets are then discussed. We proceed to define a rather broad and very important class of analytic functions, namely the class of Pfaffian functions, first introduced by Khovanskii, which includes many well known analytic functions such as polynomials, and the elementary functions on appropriate subsets of their domain of definition. Semianalytic and subanalytic sets determined by Pfaffian functions are to be called semi- and sub-Pfaffian sets, respectively; these are the central objects of our study in this thesis. After their introduction, we examine some of the finite geometrical and topological characteristics that they excibit, and state relevant results found in the existing literature. We end this Chapter by considering subanalytic sets defined over elementary extensions of the reals, which contain infinitesimal elements. The existence of such elements play a crucial role in the developement of our main construction.

3.1 Semialgebraic sets and Quantifier Elimination for Real Closed Fields

Real algebraic sets are solutions of polynomial equalities over the real field $\mathbb{R}$. Due to the fact that $\mathbb{R}$ is not algebraically closed, the class of real algebraic sets excibits rather different geometrical behaviour to that of its complex counterpart (see [BCR98, Chapter 3] or [BR90, Chapter 3]). For instance, irreducible real algebraic sets may not be connected and the subsets of their non-singular points need not be dense or connected (e.g., the cubic curve in $\mathbb{R}^2$ defined by the equation $X^2 + Y^2 - X^3 = 0$ has two connected components: an isolated point at the origin and a smooth analytic line; and the curve in $\mathbb{R}^2$ defined by the equation $XY - 1 = 0$ is non-singular and has two connected components).

The class of real algebraic sets is not closed under polynomial maps, for example the projection of the unit circle defined by the equation $X_1^2 + X_2^2 - 1 = 0$ onto the X_1-axis is the closed interval $[0, 1]$ which obviously is not a real algebraic set. As we have allready remarked in the previous chapter, Tarski-Seidenberg principle implies that semialgebraic sets in $\mathbb{R}^n$ (that is, finite Boolean combination of solutions of real polynomial equalities and inequalities in n variables) are precisely the subsets of $\mathbb{R}^n$ definable by some first-order $\mathcal{L}_{or}$-formulas with parameters from $\mathbb{R}$. It is not difficult to show that semialgebraic sets actually comprise the smallest class of subsets of $\mathbb{R}^n$, which contains the algebraic sets and which is closed under linear projections.

The remarkable stability properties of semialgebraic sets together with the decidability of the theory $Th(\mathcal{R})$ of the real ordered field make semialgebraic sets the primary object of study in the area of algorithmic real algebraic geometry and its many applications (see for example [Mis93]).

Next in this section we attempt to give a brief historical overview of some complexity highlights regarding quantifier elimination (QE) results for the first order theory of the reals. This is followed by a more detailed exposition of Collins' quantifier elimination method based on his cylindrical algebraic decomposition algorithm [Col75].

Throughout this section we consider the $\mathcal{L}_{or}$-structure $\mathcal{R} = (\mathbf{R}, +, \cdot, -, 0, 1, <)$, where $\mathbf{R}$ is a fixed real closed field. We define the complexity of an algorithm to be the number of arithmetic operations and comparisons it requires in the domain $\mathbf{R}$.

3.1.1 Brief Overview of QE results

Let $F = \{f_1, \ldots, f_s\}$ be a family of s polynomials in $n + m$ variables, each of degree at most d, with coefficients in $\mathbf{R}$.

Denote by $\Phi(Y)$ the first order $\mathcal{L}_{or}$-formula in prenex form

$$(Q_1 X^{(1)}) \cdots (Q_{p-1} X^{(p-1)})(Q_p X^{(p)}) \Theta(Y, X^{(1)}, \ldots, X^{(p)}), \tag{3.1}$$

where

- $Q_1, \ldots, Q_p$ are alternating existential and universal quantifiers, i.e, $Q_i \in \{\exists, \forall\}$ and $Q_i \neq Q_{i+1}$,
- $Y = (Y_1, \ldots, Y_m)$ is a block of m free variables,
- $X^{(i)} = (X_1^{(i)}, \ldots, X_{n_i}^{(i)})$ is a block of n_i variables, $\sum_{1 \leq i \leq p} n_i = n$, and
- $\Theta(Y, X^{(1)}, \ldots, X^{(p)})$ is a quantifier-free Boolean formula with atomic formulas of the form

$$f_j(Y, X^{(1)}, \ldots, X^{(p)}) \, \sigma \, 0, \quad 1 \leq j \leq s, \quad \sigma \in \{<, =, >\}.$$

We note that every first-order formula can be tranformed into an equivalent one in prenex form in a finite sequence of steps (see, for example [Mis93, page 356]).

If $m = 0$ then only the bounded variables $X^{(1)}, \ldots, X^{(p)}$ occur in Φ, that is Φ is a sentence of the first-order theory $Th(\mathcal{R})$ of $\mathcal{R}$, expressing some statement (true or false). The general decision problem for the first-order theory of the reals is to determine whether a given sentence Φ is true or not in $\mathcal{R}$. If in addition, $p = 1$, and $Q_1 = \exists$, i.e., the only quantifier present in Φ is an existential quantifier, then this problem is called the decision problem for the existential theory of the reals and is equivalent to the problem of deciding emptiness of a given semialgebraic set over $\mathbf{R}$.

If $m \geq 0$ then $\Phi(Y)$ may contain free variables Y. When specific values are substituted for the free variables the formula becomes a sentence. For some values of Y, $\Phi(Y)$ may be true in $\mathcal{R}$, but for some others false. The quantifier elimination problem is to construct a quantifier-free formula $\Psi(Y)$ having the same solutions as $\Phi(Y)$, that is, for any $z \in \mathbf{R}^m$, $\Phi(z)$ is true in $\mathcal{R}$ if and only if $\Psi(z)$ is true in $\mathcal{R}$. A familiar example of this is the equivalence of the $\mathcal{L}_{or}$-formula

$$\exists X \left(Y_1 X^2 + Y_2 X + Y_3 = 0\right)$$

to the quantifier-free $\mathcal{L}_{or}$-formula

$$\left((Y_1 \neq 0) \wedge (Y_2^2 - 4Y_1Y_3 \geq 0)\right) \vee \left((Y_1 = 0) \wedge \left((Y_2 \neq 0) \vee (Y_3 = 0)\right)\right).$$

Any algorithm that solves the quantifier elimination problem for the reals, solves in particular (when applied to a sentence) the general decision problem. In fact, it is often the case, that a method for performing quantifier elimination is based on a procedure for solving the general decision problem which in turn is built on a decision method for the existential theory of the reals.

Tarski was the first one to prove the decidability of the first order theory of the reals. He actualy constructed an algorithm for performing quantifier elimination [Tar48], however the complexity of his algorithm is not elementary recursive.

Efforts of Seidenberg [Sei54] and Cohen [Coh69] resulted in alternative real quantifier elimination methods, both of which apparently with worst-case running time similar to that of Tarski's method.

The first algorithm for quantifier elimination with a better worst-case running time was given by Collins [Col75]. He proved a doubly exponential in the number of variables $n+m$ upper bound on the running time of his method which was based on Cylindrical Algebraic Decomposition (CAD). We present a more detailed account of Collin's CAD method at the end of this section.

Around the same time, Wüthrich [Wüt76] constructed independently another quantifier elimination method using the same technique of cylindrical cell decomposition, with a similar upper bound on its complexity.

In the recent years a lot of attention has been drawn to this area, and new techniques were developed to attack these problems.

Grigoriev and Vorobjov [GV88] became the first to solve the decision problem for the existential theory of the reals in time singly exponential in the number of variables n. Grigoriev [Gri88] managed to extend this algorithm to a general decision method with complexity doubly exponential only for the number p of quantifier alternations and for a fixed p, singly exponential in the number of variables n.

The work of Canny [Can89] is also worth mentioning, because of the complexity improvements he achieved for the problem of deciding the existential theory of the reals, as well as his major contribution in establishing a connection with the area of robotics.

Building on ideas of Ben-Or, Kozen and Reif [BOKR86], Fitchas, Galligo and Morgenstern [FGM87] constructed a new quantifier elimination algorithm with complexity doubly exponential in the number of variables $n + m$. Quantifier elimination methods with similar or better complexity estimates were designed by Heintz, Roy and Solerno [HRS90] and Renegar [Ren92].

In 1996, Basu, Pollack and Roy [BPR96], presented an algorithm for solving both problems, with the best up-to-date complexity bounds. For the general decision problem the complexity of their algorithm is

$$s^{\Pi_i(n_i+1)}d^{\Pi_i O(n_i)}.$$

For the quantifier elimination problem their algorithm has complexity

$$s^{(m+1)\Pi_i(n_i+1)}d^{m\Pi_i O(n_i)}.$$

The algorithm produces a quantifier free formula of the form

$$\Psi(Y) = \bigvee_{1\le i\le I}\bigwedge_{1\le j\le J_i}(h_{i,j}(Y)\sigma_{i,j}0),$$

where $\sigma_{i,j} \in \{<,=,>\}$, $h_{i,j}(Y)$ are polynomials in the variables Y, of degree at most $d^{\Pi_i O(n_i)}$, and

$$I \le s^{(m+1)\Pi_i(n_i+1)}d^{m\Pi_i O(n_i)}, \qquad J_i \le s^{\Pi_i(n_i+1)}d^{\Pi_i O(n_i)}.$$

Independent results of Weispfenning [Wei88] and Davenport and Heintz [DH88] show that real quantifier elimination is inherently hard for some problem classes, and that the doubly exponential dependence on the number p of quantifier alternations of the bound on the degree of the polynomials $h_{i,j}$ can not be improved in the worst case.

More recently, Weispfenning [Wei98] introduced yet another quantifier elimination procedure based on comprehensive Grobner Bases [Wei92] and multivatiate real root counting [PRS93]; no asymptotic analysis for this method has been carried out.

3.1.2 Collins' QE method based on CAD algorithm

In what follows we denote by $\mathcal{I}_n$ the ring $\mathbf{R}[X_1,\dots,X_n]$ of polynomials in n variables with coefficients from a real closed field $\mathbf{R}$.

Let $A \subset \mathbf{R}^n$ and $f \in \mathcal{I}_n$. Then f is *invariant* on A if one of the following conditions hold: for all $\alpha \in A$, either $f(\alpha) > 0$, or $f(\alpha) < 0$, or $f(\alpha) = 0$. That is to say, f vanishes everywhere on A or nowhere. The set $F = \{f_1,\dots,f_s\} \subset \mathcal{I}_n$ is *invariant* on A if each f_i is invariant on A. In this case we also say that A is *F-invariant.*

A *Cylindrical Algebraic Decomposition* (**CAD**) of $\mathbf{R}^n$ is a cylindrical cell decomposition of $\mathbf{R}^n$ in which all of its cells are semialgebraic sets.

Every cell in a CAD can be given an index. The *index* of a cell of $\mathbf{R}^k$ is a k-tuple of positive integers. The cells in a stack are given consecutive positive integers in accordance with their position in the stack. For example, the cell (2,5) is the 5^{th} cell from the bottom, in the stack constructed in the cylinder over the 2^{nd} cell (from the left) in a CAD $\mathcal{D}_1$ of $\mathbf{R}^1$.

The input of the algorithm is a set $F = \{f_1, \ldots, f_s\} \subset \mathcal{I}_n$ of polynomials over $\mathbf{R}$ in n variables. The algorithm begins by computing another set $PROJ(F) \subset \mathcal{I}_{n-1}$ such that for each $PROJ(F)$-invariant CAD $\mathcal{D}_{n-1}$ of $\mathbf{R}^{n-1}$ there is an F-invariant CAD $\mathcal{D}_n$ of $\mathbf{R}^n$ which induces $\mathcal{D}_{n-1}$. Then the algorithm calls itself recursively on $PROJ(F)$ to get such $\mathcal{D}_{n-1}$. Finally $\mathcal{D}_{n-1}$ is extended to $\mathcal{D}_n$. If $n = 1$ an F-invariant CAD of $\mathbf{R}^1$ is directly constructed. The sign of a polynomial from F in a cell of the decomposition can be determined by computing its sign at a sample point belonging to the cell.
The general algorithm consists of three phases:

Projection: computing successive sets of polynomials in $n-1$, $n-2, \ldots, 1$ variable(s); the zeros of each set contain projections of the "significant" points of the zeros in the next higher dimensional space (which include singularities, selfcrossings, cusps, isolated points and points at which tangent is vertical).

Base: constucting a decomposition of $\mathbf{R}^1$: set of points plus finite and two infinite intervals bounded by them.

Extension: successive extensions of the decomposition of $\mathbf{R}^{r-1}$ to a decomposition of $\mathbf{R}^r$, $2 \leq r \leq n$, including sample points construction.

Next we say a few more words about each phase.
Projection Phase.
Key to this phase is to define a map $PROJ^k$, $0 \leq k \leq n-1$, which takes a subset of $\mathcal{I}_n$ to a subset of $\mathcal{I}_{n-k}$:

$$PROJ^0(F) = F = F^{(n)} \subset \mathcal{I}_n \qquad \&$$

$$PROJ^k(F) = PROJ(PROJ^{k-1}(F)) = PROJ(F^{(n-k+1)}) = F^{(n-k)} \subset \mathcal{I}_{n-k},$$

and to prove that it has the following desired property: any $PROJ^k(F)$-invariant CAD of $\mathbf{R}^{n-k}$ is induced by some $PROJ^{k-1}(F)$-invariant CAD of $\mathbf{R}^{n-k+1}$ (suffices to show that over any semialgebraic $PROJ^k(F)$-invariant cell in $\mathbf{R}^{n-k}$, there exists an $PROJ^{k-1}(F)$-invariant algebraic stack).

Main idea of this phase is to find semialgebraic subsets of $\mathbf{R}^{n-1}$ over which the given polynomials in F have a constant number of real roots (this will ensure the cylindrical arrangement of cells). The following notion encaptures this. We say that a polynomial $g \in \mathcal{I}_n$ is *delineable* on a set $A \subset \mathbf{R}^{n-1}$ if the portion of the zeroset $Zer(g)$ of g lying in the cylinder $Z(A) \subset \mathbf{R}^n$ over A consists of m disjoint sections of $Z(A)$, for some $m \geq 0$. Moreover, we say that g is *identically zero* on A if $g(\alpha, X_n)$ is the zero polynomial, for all $\alpha \in A$. Denote by $\mathcal{S}(g, A)$ the g-invariant stack over A determined by the continuous functions whose graph make up $Zer(g) \cap Z(A)$.

The way of defining $PROJ$ must make sure that for any $PROJ(F)$-invariant subset $A \subset \mathbf{R}^{n-1}$, the following two conditions hold:

1. Each $f_i \in F$ is either delineable or identically zero on A.
2. The sections of $Z(A)$ belonging to zerosets of different functions f_i, f_j are either disjoint or identical.

Notice that if the above conditions are satisfied, then clearly we have an F-invariant stack over $A \subset \mathbf{R}^{n-1}$, namely $\mathcal{S} = \mathcal{S}(\prod_{j \in J} f_j, A)$, where J is a finite set of indices such that if $j \in J$ then f_j is not identically zero on A. Moreover one can prove that if A is a semialgebraic subset of $\mathbf{R}^{n-1}$ then $\mathcal{S}$ is also semialgebraic. This can be achieved by showing that each cell C_k, $(k \geq 1)$ in $\mathcal{S}$ is semialgebraic by producing a defining formula involving "projection" polynomials (i.e., polynomials in $PROJ^{(r)}(F)$, for some positive values of r) stating that C_k is the k^{th} cell from the bottom in the cylindrical arrangement of the stack $\mathcal{S}$.

Without getting into details we note that the projector operator

$$PROJ^k : \mathcal{I}_n \longrightarrow \mathcal{I}_{n-k}, \qquad 1 \leq k \leq n-1,$$

consists of

- Coefficients of $f \in F^{(n-k+1)} = PROJ^{k-1}(F)$ (as a polynomial in X_{n-k+1});
- Discriminant of $f \in F^{(n-k+1)}$ w.r.t. X_{n-k+1};
- Resultants of each pair of polynomials $f, g \in F^{(n-k+1)}$ w.r.t. X_{n-k+1}.

Hence, polynomials in $n-k$ variables over $\mathbf{R}$ which belong in the collection

$$F^{(n-k)} = PROJ^k(F) = PROJ(F^{(n-k+1)}) \subset \mathcal{I}_{n-k}$$

characterise the maximal connected $F^{(n-k+1)}$-delineable subsets of $\mathbf{R}^{n-k}$ (for the full details see [Col75, ACM84a]).

It is computationally advisable to begin each stage k $(1 \leq k \leq n-1)$ of the projection phase by making all polynomials $h_i, h_j \in F^{(n-k+1)}$:
(i) squarefree (sufficient to compute greatest common divisors GCD (h_i, h_i') of h_i and its derivative h_i' and set $g_i = h_i/(h_i, h_i')$, and
(ii) relatively prime (sufficient to compute GCD $h_{i,j} = (h_i, h_j)$ for all pairs (i, j), $i \neq j$ and set $g_i = h_i/h_{i,j}, g_j = h_j/h_{i,j}$).

Base Phase.
Let $F^{(1)} = PROJ^{n-1}(F) \subset \mathcal{I}_1$ be a set of monovariate polynomials. Construct $G = \prod g_j$, with $0 \neq g_j \in F^{(1)}$. The real roots $\alpha_1 < \ldots < \alpha_t$ of G will be the 0-cells of $\mathcal{D}_1$, and the intervals between them plus the two semi-infinite intervals, the 1-cells. We can determine α_k's by isolating the real roots of G . It is now a trivial task to write down the indices of the $2t+1$ cells of $\mathcal{D}_1$. We can also choose sample points for each cell; all sample points will be algebraic numbers – for the 0-cells we do not have much choice but for the 1-cells we can pick rational numbers.
Extension Phase.
First consider extension of the CAD $\mathcal{D}_1$ of $\mathbf{R}^1$ to a CAD $\mathcal{D}_2$ of $\mathbf{R}^2$. In the projection phase we have computed a set of bivariate polynomials

$$F^{(2)} = PROJ^{n-2}(F) \subset \mathcal{I}_2.$$

Let C be a cell of $\mathcal{D}_1$ with index $\mathcal{J}_C$ and sample point $\alpha_C \in C$. Let $F_C^{(2)}$ be the product of all non-zero $g(\alpha_C, X_2)$, where $g \in F^{(2)}$. We isolate the real roots of $F_C^{(2)}$; this determines $\mathcal{S}(F^{(2)}, C)$, the stack constructed over C: β is a root of $F_C^{(2)}$ if and only if (α_C, β) lies on a section of $\mathcal{S}(F^{(2)}, C)$. Use $\mathcal{J}_C$ and α_C and the isolating intervals for the roots of F_C^2 to construct cell indices and sample points for the sections and sectors of $\mathcal{S}(F^{(2)}, C)$. Note that it is possible to construct each sample point as some vector of algebraic numbers over a single simple algebraic extension of the rationals.

Extension from $\mathbf{R}^{i-1}$ to $\mathbf{R}^i$ for $3 \leq i \leq n$ follows the same procedure. Algorithmic sub-procedures for dealing with algebraic numbers, or calculations in extension fields can be found in [Col75, ACM84a].

A more geometrical way of thinking the different steps in the extension phase is that of drawing vertical lines over each sample point of the lower dimensional decomposition and then calculating the intersections between these lines with the zeroset of the next higher dimensional set of "projection" polynomials.

As a summary we give:

Theorem 3.1.1 (CAD Algorithm). *Given a finite set of s multivariate polynomials $F \subset \mathbf{R}[X_1, \ldots, X_n]$ each of degree at most d, we can effectively construct the following:*

1. *An F-invariant cylindrical algebraic decomposition $\mathcal{D}_n$ of $\mathbf{R}^n$ into semialgebraic connected cells. Each cell $C \in \mathcal{D}_n$ is homeomorphic to $\mathbf{R}^l$ for some $0 \leq l \leq n$.*

2. *A sample algebraic point α_C in each cell $C \in \mathcal{D}_n$ and defining polynomials for each sample point α_C.*

3. *A list of indices of the cells comprising $\mathcal{D}_n$.*

4. *A quantifier-free defining formula Φ_C for each cell $C \in \mathcal{D}_n$.*

The worst-case running time of the algorithm turns out to be $(sd)^{2^{O(n)}}$. The number of cells in the decomposition and the degrees of the polynomials involved in their definition do not ecxeed $d^{2^{O(n)}}$.

We say that the set of defining polynomials $F \subset \mathcal{I}_n$ is *well-based* if the following condition holds:

$$(\forall \omega \in \mathbf{R}^{n-k-1})\ (\forall f_i \in PROJ^k(F))\ \left[f_i(\omega, X_{n-k}) \neq 0\right], \quad 0 \leq k \leq n-1.$$

Resulting CAD is then said to be well-based.

Remark 3.1.2. *Given $F \subset \mathcal{I}_n$, there is always a linear change of coordinates that will give us a system of polynomials that are well-based (any random change of coordinates will almost surely do that).*

It can be shown, e.g. [Mis93, §8.6.9], that if the decomposition $\mathcal{D}_n$ produced by the CAD algorithm is well based then $\mathcal{D}_n$ has the additional property that the closure $\overline{C_i}$ of each bounded cell C_i is a union of some cells C_j's that belong to $\mathcal{D}_n$, i.e., $\overline{C_i} = \bigcup_j C_j$.

Collin's algorithm is a very powerful tool for determining many geometrical and topological characteristics of problems expressible over the reals. For example, in [SS83], the CAD method is used in order to calculate the ranks of the homology groups of a semialgebraic set. The CAD algorithm itself does not provide us with any (non-trivial) information regarding cell adjacencies. Modifications of this method which result to the calculation of cell adjacencies for $n \leq 3$, appeared in [ACM84a, ACM88].

We now turn our attention at Collin's Quantifier Elimination method for the first order theory $Th(\mathcal{R})$ of the reals, based on the CAD conctruction.

An admissible ordering of the variables is a partial ordering satisfying the following:

1. bound variables are greater than free variables;

2. the ordering of the bound variables agrees with the ordering of the quantifiers in the input formula.

Given such an admissible ordering, variables are eliminated in descending order (in other words bound variables are eliminated before the free variables).

Let $F \subset \mathcal{I}_n$ be the set of all polynomials occuring in a given first-order formula Ψ over the reals in prenex form $\big($cf (3.1)$\big)$ having Θ as its quantifier-free part. We can then decide on which cells of the derived decomposition Θ is true, by evaluating the polynomials in F over all sample points. The cylindrical arrangement of the cells enables us to treat universal and existential quantifiers like conjunctions and disjunctions and thus determine cells in the induced decomposition of the space of free variables for which Ψ holds true. Each cell is described by some quantifier-free formula; by combining these defining formulas of all cells in which Ψ is true, we can obtain a quantifier-free formula having the same solutions as Ψ. This is the so called Solution Formula Construction phase of Collins' QE method and it was studied extensively by Brown in his doctoral thesis [Bro99].

Over the years, Collin's work has been the center of extensive research (for related bibliography see [CJ98]) and many advances have been made towards improving the efficiency of his CAD-based quantifier elimination method. Results in this direction emerged from, among others, further work of Collins himself and some of his research students – McCallum [McC85, McC88] (reduced pojection operator), Arnon [Arn81, Arn88] (idea of cell clustering), Hong [Hon90, CH91] (partial CAD), and more recently Browns [Bro01] (an even better projection operator). Different variants of Collins' algorithm have been successfully implemented (Arnon, Hong) and proved to be a valuable tool for attacking some non-trivial problems in areas such as constraint solving, optimization, computational geometry, control theory and stability, automatic theorem proving and robotics (for more details, see for example [Stu99]).

3.2 Semianalytic and subanalytic sets

The theory of semianalytic and subanalytic sets has its origins in the work of Łojasiewicz [Łoj64, Łoj65], Gabrielov [Gab68], Hironaka [Hir73] and Hardt [Har75]. In what follows we introduce the main definitions and present some of the basic characteristics of these sets.

Definition 3.2.1. *We say that $S \subset \mathbb{R}^n$ is semianalytic if for all $x \in \mathbb{R}^n$ there is an open neighbourhood U of x such that $S \cap U$ is a finite Boolean combination of sets $\{x \in U : f(x) = 0\}$ and $\{x \in U : g(x) > 0\}$, where f, g are analytic functions on U.*

The class of semianalytic sets was first introduced by Łojasiewicz [Łoj64, Łoj65] who established many of their basic finiteness properties, including the following (see also [BM88]):

- any semianalytic set locally has a finite number of connected components (in the case of a relatively compact semianalytic set this result applies globally);
- the components, boundary and interior of a semianalytic set are semianalytic;
- (Łojasiewicz's inequality) if $W \subset \mathbb{R}^n$ and $f, g : W \longrightarrow \mathbb{R}$ are semianalytic functions with compact graphs and $f^{-1}(0) \subset g^{-1}(0)$, then there exist $c, r > 0$ such that for all $x \in W$, $|f(x)| \geq c|g(x)|^r$;

- existence of (Whitney) stratifications for semianalytic sets; and
- existence of triangulations for semianalytic sets.

Thus, semianalytic sets excibit (locally) similar behaviour to that of semialgebraic sets. The class consisting of semianalytic sets was extended by Łojasiewicz to include also images of its members under relatively proper real analytic maps.

Definition 3.2.2. *A subset $W \subset \mathbb{R}^n$ is subanalytic if each point $x \in \mathbb{R}^n$ admits a neighbourhood U such that $W \cap U$ is a projection of a relatively compact semianalytic set.*

According to a result of Łojasiewicz's [Łoj64], if $S \subset \mathbb{R}^n$ is a subanalytic set then S is semianalytic provided that either $dim(S) \leq 1$ or $n \leq 2$. But for $n > 2$, unlike semialgebraic sets, Tarski-Seidenberg principle may fail for semianalytic sets. The following example due to Osgood [Osg16] shows that the projection of even a compact semianalytic set defined using only polynomials and the exponential function may not be semianalytic: the set determined in $\mathbb{R}^3$ by the existential expression

$$\exists u(y > 0 \wedge u \leq 1 \wedge uy = x \wedge z = ye^u) \tag{3.2}$$

cannot be defined by an expression without quantifiers.

Gabrielov's main theorem in [Gab68] that the complement of a subanalytic set is subanalytic allowed the extension to the subanalytic setting of the finiteness results stated above for semianalytic sets (see e.g [BM88]). From the point of view of mathematical logic this result asserts the possibility of "bounded quantifier simplification": any given expression involving real analytic equations and inequalities with existential and universal quantifiers can be replaced with an equivalent existential expression, provided that the variables are bounded and they do not approach the boundary of the domain of definition of the functions.

Denote by $\mathcal{R}_{An}$ the expansion of the real ordered field $\mathbb{R}$ by the collection An of all restricted analytic functions, that is functions of the kind $\hat{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$, for each n, defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,1]^n, \\ 0 & \text{otherwise,} \end{cases}$$

where $f : U \longrightarrow \mathbb{R}$ is an analytic function in some open neighbourhood U of the closed unit cube $[0,1]^n \subset \mathbb{R}^n$. Let $\mathcal{L}_{An}$ denote the language of the structrure $\mathcal{R}_{An}$.

The example above (Osgood) shows that $\mathcal{R}_{An}$ does not admit quantifier elimination. An equivalent formulation of Gabrielov's complement theorem is that the structure $\mathcal{R}_{An}$

is model-complete; this combined with finiteness results of Łojasiewicz [Łoj64] establish the o-minimality of $\mathcal{R}_{An}$.

Using Weierstrass preparation theorem and Tarski's quantifier elimination theory for semialgebraic sets, Denef and van den Dries [DvdD88] were able to give another proof of this result; moreover they show that the structure R_{An} has quantifier elimination provided a binary function symbol D for restricted division:

$$D(x,y) = \begin{cases} x/y & \text{if } |x| \leq |y| \leq 1,\ y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

is added to the language $\mathcal{L}_{An}$.

Van den Dries, Macintyre and Marker proved in [vdDMM94] that the expansion $\mathcal{R}_{An,exp}$ of the real field by restricted analytic functions and the (unrestricted) exponential with language $\mathcal{L}_{An,exp}$ is o-minimal. They went on to show that this structure actually admits quantifier elimination if we add *log* to the language.

In [Gab96], Gabrielov deals with the question whether is possible to identify any subclasses of the restricted analytic functions for which the complement of a subanalytic set described by functions from this class is again subanalytic described by functions from the same class.

Consider a subcollection $\mathcal{F}$ of the family An of restricted analytic functions, which is closed under addition, multiplication and taking partial derivatives.

Definition 3.2.3. *A set $S \subset I^n \subset \mathbb{R}^n$ is called $\mathcal{F}$-subanalytic if it is a subanalytic set with the analytic functions involved in its defining formula belonging to $\mathcal{F}$.*

Recall that the *closure* $cl(S)$ of a subanalytic set S in an open domain G is an intersection with G of the usual topological closure of S:

$$cl(S) = \{x \in G \ : \quad \forall \varepsilon > 0 \, \exists z \in S \, (||x - z|| < \varepsilon)\}.$$

The set $\partial S = cl(S) \setminus S$ is the frontier of S (within G) and $\tilde{S} = G \setminus S$ its complement in G.

Lemma 3.2.4. *Let $S \subset I^n$ be a $\mathcal{F}$-semianalytic set. Then $cl(S)$ and ∂S are also $\mathcal{F}$-semianalytic.*

Proof. This is Lemma 1 in [Gab96]. □

Based on finiteness properties of semianalytic sets, Gabrielov was able to give an affirmative answer to the above question in the case when the restricted analytic functions involved in the definition of the input subanalytic set belong in $\mathcal{F}$.

Theorem 3.2.5. *Let $S \subset I^n$ be a $\mathcal{F}$-semianalytic set and let $W = \rho_k(S) \subset I^{n-k+1}$, $1 \leq k \leq n$, where ρ_k denotes the projection map omiting the first $k-1$ coordinates. Then $\widetilde{W} = I^{n-k+1} \setminus W$ is $\mathcal{F}$-subanalytic.*

Proof. This is Theorem 1 in [Gab96]. □

The theorem follows from the existence of a cylindrical cell decomposition D of the unit cube I^n compatible with S, provided that each cell of D is defined by an existential formula involving functions from the algebra $\mathcal{F}$. Indeed in this case, by the definition of a cylindrical cell decomposition, $D = D^{(0)}$ induces a cylindrical decomposition $D^{(k-1)}$ of the cube $I^{n-k+1} = \rho_k(I^n) = \{X_1 = \cdots = X_{k-1} = 0\} \subset \mathbb{R}^{n-k+1}$ compatible with W. So $\widetilde{W}$ can be identified with a finite union of some cells of $D^{(k-1)}$ and thus is $\mathcal{F}$-subanalytic.

A crucial step in Gabrielov's proof of this theorem is to establish the existence of (weak) stratifications of $\mathcal{F}$-semianalytic sets with the property that each stratum is also $\mathcal{F}$-semianalytic [Gab96, Lemma 2].

In Chapter 5 we present a new method for obtaining such cylindrical cell decompositions which actually does not require the existence of any stratification results. As a consequence, an alternative more elementary proof of Theorem 3.2.5 can be derived.

Denote by $\mathcal{R}_{\mathcal{F}}$, the expansion of the real ordered field by functions in $\mathcal{F}$, with language $\mathcal{L}_{\mathcal{F}}$.

Theorem 3.2.6. *The structure $\mathcal{R}_{\mathcal{F}} = (\mathbb{R}, +, \cdot, -, 0, 1, <, \mathcal{F})$ is model-complete and o-minimal.*

Proof. The model-completeness of $\mathcal{R}_{\mathcal{F}}$ is a direct consequence of Theorem 3.2.5, while its o-minimality follows from the o-minimality of $\mathcal{R}_{An}$. □

3.3 The Pfaffian setting

Pfaffian functions are solutions of certain triangular systems of first order partial differential equations with polynomial coefficients (see Definition 3.3.1). This special class of trancentental functions, which was first introduced by Khovanskii [Kho80, Kho91], contains many important members such as polynomials, the exponential and logarithmic functions and trigonometric functions in bounded domains. Khovanskii proved that in the real domain the number of non-degenerate solutions of a system of Pfaffian equations is finite and that it admits an explicit bound in terms of the format of the Pfaffian functions involved. He was able to show that this in turn implied finiteness results for many geometrical and topological characteristics of semi-Pfaffian sets (that is, semianalytic sets determined by Pfaffian functions) such as the number of their connected components or more general the sum of their Betti numbers.

3.3.1 Basic definitions and Khovanskii's bound

Definition 3.3.1. *(See [Kho80, Kho91] , and [GV95a].) A* Pfaffian chain *of the order $r \geq 0$ and degree $\alpha \geq 1$ in an open domain $G \subset \mathbb{R}^n$ is a sequence of real analytic functions $f_1, \dots, f_r$ in G satisfying Pfaffian equations*

$$df_j(X) = \sum_{1 \leq i \leq n} g_{ij}(X, f_1(X), \dots, f_j(X)) dX_i \tag{3.3}$$

for $1 \leq j \leq r$. Here each $g_{ij}(X, Y)$ is a polynomial with real coefficients in $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_j)$, of degree not exceeding α. The system (3.3) is triangular in the sense that g_{ij} does not depend on f_k for $k > j$. A function

$$f(X) = P(X, f_1(X), \dots, f_r(X))$$

where $P(X, Y_1, \dots, Y_r)$ is a polynomial over $\mathbb{R}$ of degree not exceeding $\beta \geq 1$ is a Pfaffian function *of order r and degree (α, β).*

Remark 3.3.2. *The corresponding definitions appearing in [Kho91], in which the Pfaffian chains are defined as sequences of nested integral manifolds of polynomial 1-forms, are more general than ours, although both definitions give locally the same class of functions.*

Next, we consider some examples of Pfaffian functions taken from [Kho91, GV95a].

Examples 3.3.3.

1. *Pfaffian functions of order 0 and degree $(1, \beta)$ are polynomials of degree not exceeding β.*

2. *The exponential univariate function $f(X) = e^{aX}$ is a Pfaffian function of order 1 and degree $(1, 1)$ in $\mathbb{R}$, due to the equation*

$$df(X) = af(X)dX.$$

3. *The function $f(X) = 1/X$ is a Pfaffian function of order 1 and degree $(2, 1)$ in the domain $X \neq 0$, due to the equation*

$$df(X) = -f^2(X)dX.$$

4. *The logarithmic function* $f(X) = \ln(|X|)$ *is a Pfaffian function of order 2 and degree* $(2,1)$ *in the domain* $X \neq 0$, *due to the equations*

$$df(X) = g(X)dX, \qquad dg(X) = -g^2(X)dX,$$

with $g(X) = 1/X$.

5. *The polynomial* $f(X) = X^p$ *can be considered as a Pfaffian function of order* 2 *and degree* $(2,1)$ *in the domain* $X \neq 0$, *due to the equations*

$$df(X) = pf(X)g(X)dX, \qquad dg(X) = -g^2(X)dX,$$

with $g(X) = 1/X$.

6. *The function* $f(X) = tan(X)$ *is a Pfaffian function of order* 1 *and degree* $(2,1)$ *in the domain* $X \neq \pi/2 + k\pi$, *for all* $k \in \mathbb{Z}$, *due to the equation*

$$df(X) = (1 + f^2(X))dX.$$

7. *The function* $f(X) = arctan(X)$ *is a Pfaffian function in* $\mathbb{R}$ *of order* 2 *and degree* $(2,1)$, *due to the equations*

$$df(X) = g(X)dX, \qquad dg(X) = -2Xg^2(X)dX,$$

with $g(X) = (X^2+1)^{-1}$.

8. *The function* $cos(X)$ *is Pfaffian of order* 2 *and degree* $(2,1)$ *in the domain* $X \neq \pi + 2k\pi$, *for all* $k \in \mathbb{Z}$, *due to the equations*

$$cos(X) = 2f(X) - 1, \quad df(X) = -f(X)g(X)dX, \quad dg(X) = 1/2(1 + g^2(X))dX,$$

with $f(X) = cos^2(X/2)$ *and* $g(X) = tan(X/2)$.

9. *The function* $sin(X)$ *is Pfaffian of order* 3 *and degree* $(2,1)$ *in the domain* $X \neq \pi + 2k\pi$, *for all* $k \in \mathbb{Z}$, *due to the equation*

$$df(X) = g(X)dX,$$

with $g(X) = cos(X)$.

More examples of Pfaffian functions can be found in [Kho91, GV95a, Zel99, Gab02].

The set of Pfaffian functions in an open domain G is clearly, a subalgebra of the algebra of analytic functions in G, that is closed under differentiation. In addition, one can effectively estimate the complexity cost of the application of any given operation.

Lemma 3.3.4. *(See [Kho91, GV95a])*

1. *The sum (resp. product) of two Pfaffian functions, f_1 and f_2, of orders r_1 and r_2 and degrees (α_1, β_1) and (α_2, β_2), is a Pfaffian function of the order $r_1 + r_2$ and degree $(max(\alpha_1, \alpha_2), max(\beta_1, \beta_2))$ (resp. $(max(\alpha_1, \alpha_2), \beta_1 + \beta_2)$). If the two Pfaffian functions are defined by the same Pfaffian chain of order r, then the order of the sum and product is also r.*

2. *A partial derivative of a Pfaffian function of order r and degree (α, β) is a Pfaffian function of order r and degree $(\alpha, \alpha + \beta - 1)$.*

Proof. The proof of these results is straightforward and can be found in [GV95a]. □

Lemma 3.3.5. *Let G be an open domain in $\mathbb{R}^n$ and $f : G \longrightarrow \mathbb{R}$ a Pfaffian function with Pfaffian chain $f_1, \dots, f_r$ of degree (α, β). Then its Taylor expansion*

$$\check{f}_\lambda(X - z) := \sum_{k:|k| \leq \lambda} \frac{1}{k_1! \cdots k_n!} \frac{\partial^{|k|} f}{\partial X^k}(z)(X - z)^k, \tag{3.4}$$

of order λ at $z \in G$, with $|k| = k_1 + \cdots + k_n$, is a polynomial in $X, z, f_1(z), \dots, f_r(z)$ of degree $\beta + \alpha\lambda$.

Proof. This follows easily from Lemma 3.3.4, see [Gab96, Lemma1.3]. □

Using a generalization of the Rolle theorem, Khovanskii [Kho80, Kho91] proved a real analogue of Bezout's theorem for systems of Pfaffian equations. Recall that a solution $y \in \mathbb{R}^n$ of a Pfaffian system $f_1 = \cdots = f_n = 0$ is non-degenerate if the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial X_j}(y) \right)_{1 \leq i,j \leq n}$$

is non-singular.

Proposition 3.3.6. *([Kho80, Kho91]) Let $f_1, \dots, f_n$ be Pfaffian functions in an open domain $G \subset \mathbb{R}^n$ of degree at most β in a common Pfaffian chain of order r and degree α. The number of non-degenerate solutions in G of the system $f_1 = \cdots = f_n = 0$ is bounded by*

$$\beta^n O(n\beta + min(r, n)\alpha)^r 2^{r(r-1)/2}. \tag{3.5}$$

An important subclass of the Pfaffian functions which is closed under taking partial derivatives is the class of fewnomials or sparse polynomials.

Definition 3.3.7. *(See [Kho80, Kho91] and [GV95a, Gab96].) Let $X = (X_1, \dots, X_n)$ and $\mathcal{K}$ be a set of m monomials $u_1(X), \dots, u_m(X)$, where $u_i(X) = X_1^{p_{i1}} \cdots X_n^{p_{in}}$. A polynomial $f(X)$ is a $\mathcal{K}$-fewnomial of pseudodegree β if*

$$f(X) = g(X, u_1(X), \dots, u_m(X)),$$

where $g(X)$ is a polynomial of degree β in $X_1, \dots, X_n, u_1, \dots, u_m$.

Let $f(X)$ be a $\mathcal{K}$-fewnomial of pseudodegree β where $\mathcal{K}$ is a set of m monomials in X as in the definition above. Let $v_i(X) = 1/X_i$, $1 \leq i \leq n$. Then $f(X)$ is a Pfaffian function in the domain $G = \{X_1 \cdots X_n \neq 0\}$, with Pfaffian chain

$$v_1(X), \dots, v_n(X), u_1(X), \dots, u_m(X)$$

of rank $n + m$ and degree $(2, \beta)$, due to the equations

$$dv_i(X) = -v_i^2(X) dX_i, \qquad du_j(X) = u_j(X) \sum_{1 \leq i \leq n} p_{ji} v_i(X) dX_i.$$

In the case when the functions involved in Proposition 3.3.6 are polynomials, Khovanskii was able to deduce that the number of real positive roots of a system of n polynomial equations in n variables admits an upper bound in terms of the number of non-zero terms appearing in this system, independent of their degrees.

Corollary 3.3.8. *(See [Kho80, Cor 7, page 80] and [Kho91].) The number of non-degenerate roots of a polynomial system $f_1 = \cdots = f_n = 0$ lying in the positive orthant $\mathbb{R}^n_{>0}$ does not exceed*

$$(n+1)^m 2^{m(m-1)/2}, \tag{3.6}$$

where m is the number of different monomials which occur with a nonzero coefficient in a polynomial f_j.

According to Khovanskii himself [Kho80], these results "originated from unsuccessful attempts to prove Kusnirenko's conjecture for $n > 1$", and the bounds appearing above are "apparently considerably overstated". This conjecture which dates back in the 70's asserts that the upper bound in Corollary 3.3.8 can be lowered to $\prod_{1 \leq i \leq n} (m_i - 1)$, where m_i is the number of monomials appearing in the polynomial f_i. For $n = 1$ this is an immediate consequence of Descartes rule of signs (a 300+ year old result). Kusnirenko's conjecture for $n > 1$ has been open for almost 30 years, until very recently, a counterexample was produced. In the case of a system of two threenomials in two variables (i.e., $n = 2$, $m_1 = m_2 = 3$) the conjecture was predicting at most 4 nondegenerate roots in the positive quadrant, but according to Hass [Haa02], the system

$$X_1^{108} + 1.1X_2^{54} - 1.1X_2 \;=\; X_2^{108} + 1.1X_1^{54} - 1.1X_1 \;=\; 0$$

has 5 such roots.

Remark 3.3.9. *In [LRW02], the authors prove that the number of isolated roots in the positive quadrant of a system of two threenomials in two variables is at most* 5. *This improves dramatically the respective bound that can be obtained from Corollary 3.3.8 which is* 248832.

For the remaining of this thesis, unless otherwise stated, we consider only the restricted case in which Pfaffian functions are defined also on the boundary of their domain.

Definition 3.3.10. (Semi- and sub-Pfaffian set.)

1. *A set $S \subset \mathbb{R}^s$ is called* semi-Pfaffian *in an open domain $G \subset \mathbb{R}^s$ if it consists of points from G satisfying a Boolean combination of atomic equations and inequalities $f = 0, g > 0$, where f, g are Pfaffian functions having a common Pfaffian chain defined in the domain G.*

2. *Consider $I^{m+n} \subset G$, where $G \subset \mathbb{R}^{m+n}$ is an open domain, and the projection map*
$$\pi : \quad \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^n.$$
A subset $W \subset \mathbb{R}^n$ is called (restricted) sub-Pfaffian *if $W = \pi(S)$ for semi-Pfaffian set $S \subset I^{m+n}$.*

A natural notion of format for these sets was introduced by Gabrielov and Vorobjov in [GV95a].

Definition 3.3.11. (Format.) *For a semi-Pfaffian set*

$$S = \bigcup_{1 \le l \le M} \{f_l = 0, g_{l1} > 0, \ldots, g_{lJ_l} > 0\} \subset G \subset \mathbb{R}^s, \tag{3.7}$$

where f_l, g_{lj} are Pfaffian functions with a common Pfaffian chain, of order r and degree (α, β), defined in an open domain G, its format *is a quintuple (N, α, β, r, s), where $N = 1 + \sum_{1 \le l \le M}(J_l + 1)$. Let $D = \alpha + \beta$.*

For $s = m + n$ and a sub-Pfaffian set $W \subset \mathbb{R}^n$ such that $W = \pi(S)$, its format *is the format of S.*

Notation 3.3.12. *In order to simplify the notation throughout the thesis, we shall denote the system of inequalities* $(g_{l,1} > 0) \wedge \ldots \wedge (g_{l,J_l} > 0)$ *by* $(g_l > 0)$.

Khovanskii used the results of Proposition 3.3.6, regarding the bound on the number of non-degenerate positive solutions of a system of n polynomial equation in n unknows, to obtain explicit bounds on the number of connected components of (not necessarily restricted) semi-Pfaffian sets.

Proposition 3.3.13. *([Kho80, Kho91]) The number of (definably) connected components of a (possibly unrestricted) semi-Pfaffian set* S*, with the format* (N, α, β, r, s)*, does not exceed*

$$2^{r^2} s^{O(r)} (N(\alpha + \beta))^{O(r+s)}. \tag{3.8}$$

Corollary 3.3.14. *The number of (definably) connected components of a (possibly unrestricted) sub-Pfaffian set* $W = \pi(S)$*, with format* (N, α, β, r, s)*, defined by a formula having only existential quantifiers, does not exceed bound (3.8).*

Proof. This follows immediately from Proposition 3.3.13, since the number of connected components of a projection of a set is less or equal the number of connected components of the set itself. □

3.3.2 Finiteness results for the restricted case

Consider the language $\tilde{\mathcal{L}}_{RP}$ of ordered rings with parameters in $\mathbb{R}$, augmented by function symbols for all restricted Pfaffian functions of several arguments from $\mathbb{R}$ taking values in $\mathbb{R}$.

Semi-Pfaffian subsets of $\mathbb{R}^n$ are precisely the sets determined by quantifier-free $\tilde{\mathcal{L}}_{RP}$-formulas, and sub-Pfaffian subsets of $\mathbb{R}^n$ are sets determined by existential $\tilde{\mathcal{L}}_{RP}$-formulas.

Wilkie [Wil96] was the first one to prove the complement theorem for the class of restricted sub-Pfaffian sets (see also [Gab96]). He showed that the complement $I^n \setminus W$ in $I^n = \pi(I^{n+m})$ of a sub-Pfaffian set $W = \pi(S) \subset I^n$ is also sub-Pfaffian. This follows from the existence of a cylindrical cell decomposition $\mathcal{D}$ of the unit cube I^{n+m} compatible with a semi-Pfaffian set $S \subset I^{n+m}$, so that all cells in $\mathcal{D}$ are defined by existential formulas of $\tilde{\mathcal{L}}_{RP}$. As a result, subsets of $\mathbb{R}^n$ definable by arbitrary $\tilde{\mathcal{L}}_{RP}$-formulas, can actually be determined by existential $\tilde{\mathcal{L}}_{RP}$-formulas and hence are sub-Pfaffian.

Let $\tilde{\mathcal{R}}_{RP}$ denote the $\tilde{\mathcal{L}}_{RP}$-structure with underlying set $\mathbb{R}$.

Theorem 3.3.15. *The structure* $\tilde{\mathcal{R}}_{RP}$ *is model complete and o-minimal.*

Proof. The first part of the theorem is a reformulation of Wilkie's complement theorem for restricted sub-Pfaffian sets. The o-minimality of this structure follows from its model-completeness and Khovanskii's bound on the number of connected components of sub-Pfaffian sets (Corollary 3.3.14). □

The various results obtained by Gabrielov in [Gab96] hold for semianalytic and subanalytic sets defined by functions belonging to a subalgebra of analytic functions closed under differentiation. So in particular, the closure and frontier of a semi-Pfaffian set (within the unit cube) are semi-Pfaffian, any semi-Pfaffian set admits a (weak) stratification, and as we have allready discussed above, the complement (within the unit cube) of a sub-Pfaffian set is sub-Pfaffian. Of course these results also remain valid in the case when the functions defining the semi- and sub-Pfaffian sets are restricted to special subclasses of Pfaffian functions, such as the class of fewnomials.

Using a complex analogue of Rolle theorem, Gabrielov [Gab95] managed to obtain explicit estimates for the multiplicity of the intersection of complex algebraic and Pfaffian varieties. One consequence of this, was the effective evaluation by Gabrielov in the same paper, of Łojasiewicz exponents for semi-Pfaffian sets in the real domain.

These new effective estimates from [Gab95], combined with Khovanskii's bound on similar finiteness properties of semi-Pfaffian sets from [Kho80, Kho91], as well as the constructive nature of the techniques employed in the proofs of the results in [Gab96], made it possible to turn these proof-methods into algorithmic procedures with effective bounds on their complexity in terms of the format of the input semi- or sub-Pfaffian set.

In [GV95a], Gabrielov and Vorobjov constructed an algorithm (subject to the provision that an oracle deciding consistency of a system of Pfaffian equations and inequalities is given) which produces a weak stratification of a semi-Pfaffian set (i.e., a subdivision into smooth non-intersecting strata without any requirements of how these strata meet). An effective bound on the complexity of this algorithm and on the parameters of its outcome was also obtained in terms of the format of the input semi-Pfaffian set. Assuming that each oracle call has a unit cost, this bound turns out to be doubly exponential in the number of variables n, singly exponential in the maximal order r of the Pfaffian functions involved in the definition of the input semi-Pfaffian set for fixed n, and polynomial in all the rest of the parameters for fixed n and r. The exact formulation of this result can be found in [GV95a, Theorem 3]; the case when the Pfaffian functions involved in the definition of the input semi-Pfaffian set are fewnomials was also considered, and the corresponding bounds appear in [GV95a, Corollary 3].

In [Gab98], Gabrielov proved the existence of an algorithm for producing semi-Pfaffian representations of the closure and frontier of a given semi-Paffian set (within the domain of definition of the Pfaffian functions involved). The complexity of the algorithm as well as the parameters of these representations are estimated in terms of the format of the input semi-Pfaffian set.

Gabrielov was able to show [Gab98, Lemma3.1] that the estimates for the multiplicity of Pfaffian intersections from [Gab95], allow us to determine a sufficiently large natural number q, such that the problem of deciding whether a given point z belongs to the closure of a semi-Pfaffian set S, can be answered by testing whether z belongs to the closure of an auxiliary semialgebraic set $\check{S}_z$ involving Taylor expansions at z of finite order q of the Pfaffian functions present in the definition of S. Using an algebraic quantifier elimination method [BPR96] we can then obtain a semialgebraic condition involving polynomials in $X, z, f_1(z), \dots, f_r(z)$, where $f_1, \dots, f_r$ is a common Pfaffian chain of the Pfaffian functions present in S (see Lemma 3.3.5), satisfied exactly when z belongs in the closure of $\check{S}_z$. So, the set of those z for which this semialgebraic condition is satisfied is semi-Pfaffian.

Lemma 3.3.16. *Let S be a semi-Pfaffian set in an open domain $G \subset \mathbb{R}^s$, of format (N, α, β, r, s) and defined by (3.7) where $s = n + m$ and the variables are $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_m)$. There is an algorithm which produces a Boolean formula $F(X,Y)$ in disjunctive normal form with atomic Pfaffian functions such that for any fixed vector $y \in \mathbb{R}^m$ the closure $cl(S \cap \{Y = y\}) \subset \mathbb{R}^n$ coincides with $\{F(X,y)\}$. The format of $\{F(X,Y)\}$ is*

$$((Nd)^{O((s+r)s)}, \alpha, d^{O(s)}, r, s),$$

where $d = 2^{r^2}(sD)^{s+r}$ and $D = \alpha + \beta$. The complexity of the algorithm does not exceed

$$(Nd)^{O((s+r)s)}.$$

Proof. The proof of this lemma is a straightforward parameterization of the proof of Theorem 1.1 from [Gab98]. □

The corresponding bounds for the case when the Pfaffian functions involved in the definition of S are fewnomials, can be found in [Gab98, Theorem 4.1].

The result above cannot be extended for Pfaffian functions in an unbounded domain or at the boundary of their domain of definition since its proof involves Taylor expansions. It is important that we consider the frontier and closure of a semi-Pfaffian set only within its domain of definiton: in [Gab97] an example was given which implies that the unrestricted frontier and closure of a fewnomial semialgebraic set is not fewnomial.

Richardson introduced in [Ric99] the notion of "infinitesimal quantifiers":

- $(\exists X \sim 0)A$ to be read "there exists arbitrary small X so that A is true";
- $(\forall X \sim 0)A$ to be read "for all sufficiently small X, A is true",

and he showed that as a consequence of Gabrielov's work on the closure of a semi-Pfaffian set (see Lemma 3.3.16), these quantifiers can be effectively eliminated from first order formulas involving restricted Pfaffian functions (of course this is not possible for ordinary quantifiers). He proved that Whitney regularity conditions in particular, can be expressed in terms of these infinitesimal quantifiers; from this he deduced the existence of Whitney stratifications for semi- and sub-Pfaffian sets. No explicit estimates as some functions of the format of the input are given for the complexity of these constructions.

Recently Gabrielov and Vorobjov in [GV01] suggested an algorithm which produces cylindrical cell decompositions of sub-Pfaffian sets in $\mathbb{R}^n$. In particular, this algorithm finds complements to sub-Pfaffian sets, in other words eliminates one sort of quantifiers from prenex first-order formulas involving restricted Pfaffian functions. As a model of computation [GV01] uses a *real numbers machine* (Blum-Shub-Smale model) [BCSS97] equipped with an *oracle* for deciding the feasibility of any system of Pfaffian equations and inequalities. The complexity bound of this algorithm, the number and formats of cells are doubly exponential in $O(n^3)$ (assuming that each oracle call has a unit cost). The exact formulation of these bounds is given in [GV01, Section 2]). The mathematical technique employed in [GV01] is based on differential geometry, in particular an efficient smooth stratification procedure from [GV95a] is used as a subroutine.

In Chapter 6 we present a new and more elementary algorithm for producing cylindrical cell decompositions of sub-Pfaffian sets in $\mathbb{R}^n$. As in [GV01], this algorithm is conditional because we need to assume the existence of an oracle for deciding emptiness of semi-Pfaffian sets; moreover it can be realized on a version of a real numbers machine. The complexity bound of our algorithm, the number and formats of cells are doubly exponential in $O(n^2)$ (assuming that each oracle call has a unit cost). This bound, being formally incomparable with the one from [GV01], is better for a long Pfaffian chain for defining functions.

We finish this section with a remark. As we have allready mentioned, the connected components of a semianalytic set are semianalytic (see e.g [BM88]). To the best of our knowledge, the question whether is possible to obtain a semi-Pfaffian representation for each of the connected components of a semi-Pfaffian set S together with explicit estimates on the format of each such set in terms of the format of S, has not yet been resolved.

3.3.3 The real exponential field

Next we consider briefly the "unrestricted" case in which Pfaffian functions may not be defined on the boundary of their domain or may be defined in an unbounded domain. So far we have been dealing with analytic functions restricted to compact subsets of their natural domain. Unlike semialgebraic sets, in general, a semianalytic set need not remain semianalytic at infinity (that is, when $\mathbb{R}^n$ is compactified to real projective space $\mathbb{R}^n P$).

Let $\mathcal{L}_{exp}$ denote the language of ordered rings with a symbol for exponentiation and let $\mathcal{R}_{exp} = (\mathbb{R}, +, \cdot, 0, 1, <, e^x)$ be a $\mathcal{L}_{exp}$-structure.

The fact that $\mathcal{R}_{exp}$ does not admit quantifier elimination was first proved by Osgood, [Osg16] who gave an example of a subanalytic set in $\mathbb{R}^3$ defined by polynomials and the exponential function that could not be defined without quantifiers. In fact, Gabrielov gave an example in [Gab97] which implies that bounded quantifier elimination is impossible for expressions with the exponential function even if an additional operation of bounded division is allowed (compare with results in [DvdD88]).

In [Wil96], Wilkie proved the following remarkable result.

Theorem 3.3.17. *The real exponential field $\mathcal{R}_{exp}$ is model-complete and o-minimal.*

The first part of this theorem implies that for every $\mathcal{L}_{exp}$-formula $\Phi(X_1, \ldots, X_n)$, there exists $k \geq n$, and exponential polynomials

$$f_1, \ldots, f_l \in \mathbb{Z}[X_1, \ldots, X_k, e^{X_1}, \ldots, e^{X_k}],$$

such that $\Phi(X_1, \ldots, X_n)$ is equivalent to the existential formula

$$\exists X_{n+1} \cdots \exists X_k \; f_1(X_1, \ldots, X_k, e^{X_1}, \ldots, e^{X_k}) = \cdots = f_l(X_1, \ldots, X_k, e^{X_1}, \ldots, e^{X_k}) = 0.$$

Thus, any $\mathcal{R}_{exp}$-definable set can be obtained as the projection of an exponential variety. The o-minimality of $\mathcal{R}_{exp}$ follows from Khovanksii's bound (Proposition 3.3.13) on the number of connected components of quantifier-free $\mathcal{R}_{exp}$-definable sets.

Khovanskii's finiteness theorem regarding the number of connected components of exponential varieties is an important ingredient of Wilkie's actual proof of the model-completeness of $\mathcal{R}_{exp}$. In order to obtain this result Wilkie uses the model-completeness of the expansion of the real ordered field by the restricted exponential function (a particular case of the complement theorem for restricted sub-Pfaffian sets which he proves in the first part of the same paper [Wil96]), together with model-theoretic arguments to analyse large zeros of systems of exponential-algebraic equations.

Apparently, no geometric proof of this result has been established so far; explaining the behaviour of certain analytic functions at infinity is one of the most outstanding contributions of the model-theoretic point of view in subanalytic geometry.

The question whether the theory of $\mathcal{R}_{exp}$ is decidable was first raised by Tarski in [Tar48], where he proved the decidability of the real ordered field.

Using model theoretic methods, Macintyre and Wilkie [MW95] showed that the first order theory of $\mathcal{R}_{exp}$ is indeed decidable, provided that the following famous conjecture from transcendental number theory holds.

Schanuel's Conjecture for $\mathbb{R}$: If $\alpha_1, \dots, \alpha_n$ are real numbers linearly independent over $\mathbb{Q}$, then the field

$$\mathbb{Q}(\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n})$$

has transcendence degree at least n over $\mathbb{Q}$.

This conjecture is generally thought to be correct but very difficult to prove at present.

In an independent developement Richardson [Ric97] obtained a similar result (which actually follows from the work of Macintyre and Wilkie [MW95]): he showed that the zero recognition problem for a fairly broad class of complex numbers (elementary numbers) can be solved provided that the Schanuel's Conjecture (for $\mathbb{C}$) is true. The techniques he employed in his proof have a strong algorithmical flavour amd are rather different to those used in [MW95].

Although the decidability of the existential theory of the real exponential field is yet to be proven, this problem has been resolved in situations when the terms appearing in the existential $\mathcal{L}_{exp}$-formulas are of a special form.

In [Vor92], Vorobjov constructed an algorithm for deciding consistency in $\mathbb{R}^n$ of a system of equations and inequalities involving terms of the kind

$$g\big(X_1, \dots, X_n, e^{h(X_1, \dots, X_n)}\big),$$

where h is a fixed integral polynomial in variables $X_1, \dots, X_n$ and g is an arbitrary integral polynomial in variables $X_1, \dots, X_n, X_{n+1}$. If the number of these terms do not exceed k and d is a bound on the degree of the polynomials g and h, then the running time of this algorithm turns out to be at most $O(nkd)^{n^4}$.

3.3.4 Expansion of the reals by unrestricted Pfaffian functions

Denote by $\mathcal{R}_P$ the expansion of the ordered ring of the reals by total (unrestricted) Pfaffian functions, with language $\mathcal{L}_P$. In the seminal paper [Wil99], Wilkie was able to show, as a particular case of a more general theorem of the complement, that the structure $\mathcal{R}_P$ is o-minimal and thus $\mathcal{R}_P$-definable sets enjoy the "tame" properties that the o-minimality setting has to offer (see Section §2.4).

To give a precise statement of this theorem we follow closely Wilkie [Wil98].

Let $\tilde{\mathcal{R}}$ be any expansion of the real ordered field $\mathcal{R} = (\mathbb{R}, +, \cdot, -, 0, 1, <)$, with language $\tilde{\mathcal{L}}$ extending $\mathcal{L}_{or}$.

We say that a formula $\Phi(X_1, \dots, X_n, Y_1, \dots, Y_m)$ of $\tilde{\mathcal{L}}$ is *tame* if there exists a natural number N which depends only on Φ such that for any choice of $(c_1, \dots, c_m) \in \mathbb{R}^m$ the set

$$\{(\beta_1, \dots, \beta_n) \in \mathbb{R}^n : \tilde{\mathcal{R}} \models \Phi(\beta_1, \dots, \beta_n, c_1, \dots, c_m)\}$$

has at most N connected components.

Notation 3.3.18. *It is convenient to introduce a unary connective (that is actually allready definable in $\tilde{\mathcal{L}}$), denoted by C, to our language, with truth condition:*

$$\tilde{\mathcal{R}} \models (\mathcal{C})\Phi(\alpha_1, \dots, \alpha_n) \Longleftrightarrow$$

$$(\alpha_1, \dots, \alpha_n) \in cl\big(\{(\beta_1, \dots, \beta_n) \in \mathbb{R}^n : \tilde{\mathcal{R}} \models \Phi(\beta_1, \dots, \beta_n)\}\big),$$

where cl denotes here the closure in the Euclidean topology in $\mathbb{R}^n$.

Using techniques of Tarski systems and building on ideas of Charbonnel [Cha91], Wilkie proved the following theorem.

Theorem 3.3.19. *With $\tilde{\mathcal{R}}$ as above, suppose that every quantifier-free $\tilde{\mathcal{L}}$-formula is tame.*
(i) Then any formula that can be obtained from quantifier-free formulas by finitely many applications of conjuction, disjunction, existential quantification and the connective C is also tame.
(ii) In addition, if we assume that $\tilde{\mathcal{R}}$ is the expansion of $\mathbb{R}$ by infinitely differentiable real functions, then any formula of $\tilde{\mathcal{L}}$ is equivalent (in $\tilde{\mathcal{R}}$) to one of the type described above, and hence the structure $\tilde{\mathcal{R}}$ is o-minimal.

The o-minimality of $\mathcal{R}_P$ then follows from Khovankii's result (Proposition 3.3.13) regarding the tameness of quantifier-free $\mathcal{L}_P$-formulas, where $\mathcal{L}_P$ is the language of $\mathcal{R}_P$.

As a consequence, any $\mathcal{R}_P$-definable set consists of finitely many $\mathcal{R}_P$-definably connected components, each of which is $\mathcal{R}_P$-definable. Apparently, no effective explicit upper bound on the number of connected components of definable sets in this structure, can be obtained from this work (in fact, this was never intended to be one of the issues addressed in [Wil99]).

Generalizations of Wilkie's result from [Wil99] appeared in [KM99, Spe99].

Very recently, Gabrielov introduced in [Gab02] the relative closure operation on one-parametric families of semi-Pfaffian sets and was able to give a more manageable description of the definable sets in the structure $\mathcal{R}_P$ (which he called "limit sets"). A notion of complexity (which extends the Pfaffian complexity, see Definition 3.3.11) was associated to these limit sets and was employed in order to estimate the complexity of Boolean operations in this structure. In [GZ02] an explicit upper bound is given for the number of connected components of a limit set in terms of its format.

Most of the results that hold for the restricted Pfaffian case also hold for the unrestricted one as well. But unlike the restricted case there is no way that we know of, to express the closure of a quantifier-free $\mathcal{R}_P$-definable set as a quantifier-free $\mathcal{R}_P$-definable set. In other words, the closure of an unrestricted semi-Pfaffian set may not be semi-Pfaffian; the question whether or not this is possible remains an open problem. A positive answer to this would imply the model-completeness of $\mathcal{R}_P$.

3.4 Subanalytic sets over nonstandard extensions of the reals

In this last section we deal with extensions of the real ordered field $\mathbb{R}$ by infinitesimal elements. We present a digest from Robinson's nonstandard analysis, for a detailed exposition the reader is referred to the texts [Rob66, Dav77, ACH97]; shorter introductions to the subject can be found in [Buc89, Jao00]. Some applications of nonstandard analysis to other areas, such as mathematical physics, probability theory and mathematical finance theory can be found in [ACH97].

In this thesis we are mainly interested in the following structures:

- $\mathcal{R}_{An} = (\mathbb{R}, +, \cdot, -, 0, 1, <, An)$, the expansion of the real ordered field $\mathbb{R}$ by the family An of restricted analytic functions, with language $\mathcal{L}_{An}$ and
- $\mathcal{R}_{\mathcal{F}} = (\mathbb{R}, +, \cdot, -, 0, 1, <, \mathcal{F})$, the expansion of $\mathbb{R}$ by a collection $\mathcal{F}$ of restricted analytic functions closed under addition, multiplication and partial derivation, with language $\mathcal{L}_{\mathcal{F}}$.

Definition 3.4.1. *Let $\mathbb{R} \subset \mathbb{R}_k \subset \mathbb{R}_{k+1}$ be real closed fields, $z \in \mathbb{R}_{k+1}$. Then*

1. *z is infinitesimal relative to $\mathbb{R}_k$, if $|z| < w$ for all $0 < w \in \mathbb{R}_k$.*
2. *z is $\mathbb{R}_k$-finite if $|z| < w$, for some $w \in \mathbb{R}_k$.*
3. *z is infinite relative to $\mathbb{R}_k$, if $|z| > w$ for all $w \in \mathbb{R}_k$.*

There exists a sequence of ordered fields

$$\mathbb{R} = \mathbb{R}_0 \subset \mathbb{R}_1 \subset \mathbb{R}_2 \subset \cdots \subset \mathbb{R}_k \subset \cdots \tag{3.9}$$

in which the field $\mathbb{R}_{k+1}$, $k \geq 0$, contains an element $\varepsilon_k > 0$ infinitesimal relative to the elements of $\mathbb{R}_k$, such that

$$\mathcal{R}_{An}^{(0)} \preceq \mathcal{R}_{An}^{(1)} \preceq \mathcal{R}_{An}^{(2)} \preceq \cdots \preceq \mathcal{R}_{An}^{(k)} \preceq \cdots \tag{3.10}$$

where $\mathcal{R}_{An}^{(k)} = (\mathbb{R}_k, +, \cdot, -, 0, 1, <, An)$ is an $\mathcal{L}_{An}$-structure.

Let $\tilde{\mathcal{L}}_{An}^{(k)}$ denote the language $\mathcal{L}_{An}$ augmented by constant symbols for each element of $\mathbb{R}_k$, and $\tilde{\mathcal{R}}_{An}^{(k)}$ denote the $\tilde{\mathcal{L}}_{An}^{(k)}$-structure $(\mathbb{R}_k, +, \cdot, -, \{c\}_{c \in \mathbb{R}_k}, <, An)$ expanding $\mathcal{R}_{An}^{(k)}$.

It follows from chain (3.10) of elementary extensions, that the following "transfer principle" is valid: for all integers $0 \leq i < j$ and any $\tilde{\mathcal{L}}_{An}^{(i)}$-sentence Φ,

$$\tilde{\mathcal{R}}_{An}^{(i)} \models \Phi \Longleftrightarrow \tilde{\mathcal{R}}_{An}^{(j)} \models \Phi.$$

The model-completeness and o-minimality of $\mathcal{R}_{An}^{(k)}$, $k > 0$, follows from the model-completeness and o-minimality of $\mathcal{R}_{An}^{(0)} = \mathcal{R}_{An}$.

Two elements $x, y \in \mathbb{R}_j$ are said to be infinitely close relative to $\mathbb{R}_i$, $0 \leq i < j$, and we write $x \approx_i y$, if $|x - y|$ is infinitesimal relative to $\mathbb{R}_i$. It is not difficult to see that $\approx_i$ is an equivalent relation on $\mathbb{R}_j$. We define $\mu_i^j(x) = \{y \in \mathbb{R}_j : x \approx_i y\}$, the set of points of $\mathbb{R}_j$ that are infinitely close to x relative to $\mathbb{R}_i$.

Let $Fin_i(\mathbb{R}_j)$ denote the $\mathbb{R}_i$-finite points of $\mathbb{R}_j$. This set forms a subring of $\mathbb{R}_j$ with unique maximal ideal the set $\mu_i^j(0)$ of infinitesimal elements of $\mathbb{R}_j$ relative to $\mathbb{R}_i$. One can prove that if an element $z \in Fin_i(\mathbb{R}_j)$, then there exist unique elements $z' \in \mathbb{R}_i$ and $z'' \in \mu_i^j(0)$, such that $z = z' + z''$, in other words $z \approx_i z'$. In this case z' is called the standard part of z (relative to $\mathbb{R}_i$) and is denoted by $z' = \mathrm{st}_i(z)$. The map

$$\mathrm{st}_i : Fin_i(\mathbb{R}_j) \longrightarrow \mathbb{R}_i$$

is an order preserving homomorphism of $Fin_i(\mathbb{R}_j)$ onto $\mathbb{R}_i$. One can extend the map st_i (component-wise) to vectors from $\mathbb{R}_j^n$ with $\mathbb{R}_i$-finite components, and (element-wise) to $\mathbb{R}_i$-finite subsets of $\mathbb{R}_j^n$.

In the rest of this thesis we will sometimes denote the distinguished infinitesimal from the field $\mathbb{R}_k$ differently (from ε_k) and adjust the notations for the field itself and for a standard part, relative to this field, accordingly.

Let $\tilde{\mathcal{L}}_{\mathcal{F}}^{(k)}$ denote the language $\mathcal{L}_{\mathcal{F}}$ augmented by constant symbols for each element of $\mathbb{R}_k$, and $\tilde{\mathcal{R}}_{\mathcal{F}}^{(k)}$ denote the $\tilde{\mathcal{L}}_{\mathcal{F}}^{(k)}$-structure expanding $\mathcal{R}_{\mathcal{F}}^{(k)} = (\mathbb{R}_k, +, \cdot, -, 0, 1, <, \mathcal{F})$. A subset of $\mathbb{R}_k^n$ is $\tilde{\mathcal{R}}_{\mathcal{F}}^{(k)}$-definable if and only if it is $\mathcal{R}_{\mathcal{F}}^{(k)}$-definable with parameters from $\mathbb{R}_k$. Clearly, $\tilde{\mathcal{R}}_{\mathcal{F}}^{(k)} \subset \tilde{\mathcal{R}}_{An}^{(k)}$ and any $\tilde{\mathcal{L}}_{\mathcal{F}}^{(k)}$-formula is also a $\tilde{\mathcal{L}}_{An}^{(k)}$-formula.

We will say that a set $S \subset \mathbb{R}_k^n$ is [$\mathcal{F}$-semianalytic / $\mathcal{F}$-subanalytic] if S is definable by some [quantifier-free / existential] $\tilde{\mathcal{L}}_{\mathcal{F}}^{(k)}$-formula.

Suppose that the set $S \subset \mathbb{R}^n$ is defined by an $\tilde{\mathcal{L}}_{\mathcal{F}}^{(0)}$-formula Ψ. Then the same formula defines the *extension* $S^{(k)} \subset \mathbb{R}_k^n$ of S over the extension field $\mathbb{R}_k$. We will drop the upper index in $S^{(k)}$ when this does not lead to ambiguity.

In particular, when the collection $\mathcal{F}$ comprises of Pfaffian functions, we shall consider $\tilde{\mathbb{R}}_{\mathcal{F}}^{(k)}$-definable Pfaffian functions ϕ. By this we mean that there exists a Pfaffian function ϕ' defined over $\mathbb{R}$ (i.e. in the sense of the Definition 3.3.1) such that ϕ is a result of a replacement of some variables in the extension of ϕ' to $\mathbb{R}_k$ by some elements of $\mathbb{R}_k$ (sometimes we might say instead, that these functions are definable over $\mathbb{R}_k$).

Basic notions, introduced previously, can be naturally extended to a field $\mathbb{R}_k$ for $k > 0$. Thus, we shall consider semi-Pfaffian sets, sub-Pfaffian sets, cell decompositions, determined in $\mathbb{R}_k^n$ by $\tilde{\mathbb{R}}_{\mathcal{F}}^{(k)}$-definable Pfaffian functions. When this is the case we shall say that the sets and decompositions are $\tilde{\mathbb{R}}_{\mathcal{F}}^{(k)}$-definable (or definable over $\mathbb{R}_k$).

An $\tilde{\mathbb{R}}_{An}^{(k)}$-definably connected component of an $\tilde{\mathbb{R}}_{An}^{(k)}$-definable set $A \subset \mathbb{R}_k^n$, $k > 0$, may not be connected in the Euclidean topology on $\mathbb{R}_k^n$ (unlike the case $k = 0$). Nevertheless, it is true that any non-empty $\tilde{\mathbb{R}}_{An}^{(k)}$-definable set $A \subset \mathbb{R}_k^n$, $k \geq 0$, consists of only finitely many $\tilde{\mathbb{R}}_{An}^{(k)}$-definable components (this follows from the o-minimality of $\mathbb{R}_{An}^{(k)}$).

Assuming for a moment Theorem 3.3.15, which asserts the model completeness and o-minimality of the expansion $\mathcal{R}_{RP}$ of the real ordered field $\mathbb{R}$ by restricted Pfaffian functions, it is possible to extend (with the help of the tranfer principle) over to nonstandard elementary fields $\mathbb{R}_k$, $k > 0$, the finiteness results (and in particular, the bounds appearing in Corollary 3.3.14) with respect to the number of definably connected components of sub-Pfaffian sets (see e.g, [GV96, Section A.3]). The algebraic quantifier elimination method [BPR96] which was employed in the construction of Gabrielov's algorithm for computing the closure of a semi-Pfaffian subset of $\mathbb{R}^n$, works not only over $\mathbb{R}$, but over any arbitrary real closed field; the result stated in Lemma 3.3.16 is also valid over the fields $\mathbb{R}_k$, $k > 0$.

Remark 3.4.2. (*cf Notation 3.3.18*) *For any first-order formula in any given language, we denote by $\mathcal{C}\Phi(X)$, the formula defining the topological closure (within the unit cube) of the set determined by Φ, i.e.,*

$$\mathcal{C}\Phi(X) := \forall T \exists Y \left[\Phi(Y) \wedge \left((||X - Y||^2 < T^2) \vee (T = 0)\right)\right]. \qquad (3.11)$$

Suppose that the positive element $\delta \in \mathbb{R}_{k+1}$ is infinitesimal relative to $\mathbb{R}_k$. Let $W_\delta = \{\Psi_\delta\} \subset \mathbb{R}^n_{k+1}$ be a $\mathcal{F}$-subanalytic set determined by the existential $\tilde{\mathcal{L}}^{(k+1)}_{\mathcal{F}}$-formula

$$\exists Y_1 \cdots \exists Y_s \, \Phi_\delta(X, Y, \delta),$$

where Φ_δ is the quantifier-free part of Ψ_δ, in which the atomic functions involved have variables $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_s)$. Note that $W_\delta = \pi(\{\Phi_\delta\})$, where $\pi(X, Y) = X$.

Introduce a new variable Z and denote by Ψ_Z the $\tilde{\mathcal{L}}^{(k)}_{\mathcal{F}}$-formula obtained by replacing δ by Z in Ψ_δ. Then $W_Z = \{\Psi_Z\} \subset \mathbb{R}^{n+1}_k$, and variables X, Z occur freely in Ψ_Z. We can identify the sets Φ_δ and $\{\Phi_Z \wedge (Z = \delta)\}$. In Lemma A.8 of Appendix A, we show that

$$st_k\big(\{\Phi_Z \wedge (Z = \delta)\}\big) = cl\big(\{\Phi_Z \wedge (Z > 0)\} \cap \{Z = 0\}\big). \tag{3.12}$$

Note that the right hand side of the above equality is indeed $\mathcal{F}$-semianalytic due to a result of Gabrielov [Gab96, Lemma 1] in the case when the collection $\mathcal{F}$ is closed under addition, multiplication and partial derivation.

Proposition 3.4.3. *The set* $S = st_k(W_\delta) \subset \mathbb{R}^n_k$ *is* $\mathcal{F}$-subanalytic *determined by the* $\tilde{\mathcal{L}}^{(k)}_{\mathcal{F}}$*-formula*

$$\Theta := \exists Y_1 \cdots \exists Y_s \left(\mathcal{C}\big(\Phi_Z \wedge (Z > 0)\big) \wedge (Z = 0) \right).$$

Moreover $dim(S) \leq dim(W_\delta)$.

Proof.
$$\begin{aligned} S = st_k(W_\delta) &= st_k\big(\pi(\{\Phi_Z \wedge (Z = \delta)\})\big) = \pi\big(st_k(\{\Phi_Z \wedge (Z = \delta)\})\big) \\ &= \pi\big(cl(\{\Phi_Z \wedge (Z > 0)\} \cap \{Z = 0\})\big) \quad \text{by (3.12)} \\ &= \{\Theta\}. \end{aligned}$$

For a proof of the remaining part of the proposition, see [GV96, Lemma A.8]. ☐

Chapter 4

Cylindrical cell decompositions of semianalytic sets

In this Chapter we develop a new method, based on simple geometrical ideas, for constructing cylindrical cell decompositions of semianalytic sets; this method in particular, does not rely on any stratification results for semianalytic sets.

One can refer to Robinson's nonstandard analysis to justify the existence of a sequence of real closed fields $\mathbb{R}_k \subset \mathbb{R}_{k+1}$, $k \geq 0$, with $\mathbb{R}_0 = \mathbb{R}$ the field of real numbers, having the property that $\mathbb{R}_{k+1}$ contains a positive element ε_k infinitesimal relative to $\mathbb{R}_k$. Throughout this chapter we deal with the structures

$$\tilde{\mathcal{R}}_{An}^{(k)} = (\mathbb{R}_k, +, \cdot, -, \{c\}_{c\in\mathbb{R}_k}, <, An), \qquad \tilde{\mathcal{R}}_{\mathcal{F}}^{(k)} = (\mathbb{R}_k, +, \cdot, -, \{c\}_{c\in\mathbb{R}_k}, <, \mathcal{F}),$$

first introduced in Section §3.4, with languages $\tilde{\mathcal{L}}_{An}^{(k)}$ and $\tilde{\mathcal{L}}_{\mathcal{F}}^{(k)}$ respectively, where An denotes the family of restricted analytic functions and $\mathcal{F}$ is a subalgebra of An closed under taking partial derivatives. We need to assume the o-minimality of the structure $\tilde{\mathcal{R}}_{An}^{(k)}$, which follows from the o-minimality of $\mathcal{R}_{An}$ [Gab68, vdD86, DvdD88]. This implies in particular, that any $\tilde{\mathcal{R}}_{An}^{(k)}$-definable subset of $\mathbb{R}_k^n$ has a finite number of definably connected components, each of which is $\tilde{\mathcal{R}}_{An}^{(k)}$-definable. Recall that a subset of $\mathbb{R}_k^n$ is said to be $\mathcal{F}$-semianalytic if it is definable by some quantifier-free $\tilde{\mathcal{L}}_{\mathcal{F}}^{(k)}$-formula; we call projections of such sets $\mathcal{F}$-subanalytic. Clearly, for $l \geq k$, any $\tilde{\mathcal{L}}_{\mathcal{F}}^{(k)}$-formula is also a $\tilde{\mathcal{L}}_{\mathcal{F}}^{(l)}$-formula. For any set $T \subset \mathbb{R}_k^n$ definable by some $\tilde{\mathcal{L}}_{\mathcal{F}}^{(k)}$-formula Ψ_T, by abuse of notation, we will denote with the same letter its extension over $\mathbb{R}_l$, (i.e., the subset of $\mathbb{R}_l^n$ defined by the same formula Ψ_T).

Consider the closed unit cube $I^n = [0,1]^n \subset G \subset \mathbb{R}^n$, where G is some neighbourhood of I^n, and a $\mathcal{F}$-semianalytic set S defined in G. We describe a certain cylindrical cell decomposition of $I^n \subset \mathbb{R}_n^n$ compatible with $S \subset \mathbb{R}_n^n$. By definition, this decomposition induces a cylindrical cell decomposition of the projection W of S onto a subspace $I^m \subset \mathbb{R}^m_n$, $m \leq n$.

In Section 4.1, we show that this more general case can be induced from the case where S is the zeroset of a single analytic function. So, it suffices to construct a cylindrical cell decomposition of I^n compatible with $Zer(f) = \{f = 0\}$, where $f \in \mathcal{F}$ is a real analytic function in n variables, defined in the open domain G. Without loss of generality, we can also assume that for arbitrarily small $\delta > 0$ in $\mathbb{R}$, the intersection $I^n \cap Zer(f)$ is homeomorphic to $[\delta, 1-\delta]^n \cap Zer(f)$.

Next, we present a very brief and informal exposition of the method we use in order to obtain the above decomposition; this is intended to serve only as a rough approximation to the detailed description that follows in Section §4.2.

Let $\Phi^{(i)}(X)$ be a $\tilde{\mathcal{L}}^{(i+1)}_{An}$-formula, for some $i \geq 0$, where $X = (X_1, \ldots, X_n)$ and let $\omega^{(n-k)} = (\omega_{k+1}, \ldots, \omega_n)$ be any vector of I^{n-k}, $1 \leq k \leq n-1$. Denote by $\{\Phi^{(i)}(X_1, \ldots, X_k, \omega^{(n-k)})\}$ the set defined by the formula obtained after replacing in $\Phi^{(i)}(X)$ variables $X_{k+1}, \ldots, X_n$ by $\omega_{k+1}, \ldots, \omega_n$ respectively.

We define a cylindrical cell decomposition $\mathcal{D}$ of I^n which is compatible with $Zer(f)$ in two phases: the descending and the ascending one.

The descending phase consists of n steps. For any vector $\omega^{(n-1)} \in I^{n-1}$, the set

$$\Gamma^{(0)}[\omega^{(n-1)}] = Zer(f) \cap \{X_2 = \omega_2, \ldots, X_n = \omega_n\},$$

which has dimension at most 1, is the input of the first step of this phase. We can then construct a formula $\Phi^{(0)}(X)$ not depending on the specific vector $\omega^{(n-1)}$, such that $\Omega^{(0)}[\omega^{(n-1)}] = \{\Phi^{(0)}(X_1, \omega^{(n-1)})\}$ is finite and includes all the "special" points (i.e., points of local extremum, isolated points and selfcrossing – see Definition 4.2.1) of the curve $\Gamma^{(0)}[\omega^{(n-1)}]$. The input of the next step is the set $\Gamma^{(1)}[\omega^{(n-2)}] = \{\Phi^{(0)}(X_1, X_2, \omega^{(n-2)})\}$. We can think of $\Gamma^{(1)}[\omega^{(n-2)}]$ as the curve in the cube $I^n \cap \{X_3 = \omega_3, \ldots, X_n = \omega_n\}$ obtained by "stretching" along X_2, the (finite in each X_2-section) set $\Omega^{(0)}[\omega^{(n-1)}]$. Similarly, we can construct a formula $\Phi^{(1)}(X)$ not depending on the specific values of the componets of the vector $\omega^{(n-2)}$, such that the set $\Omega^{(1)}[\omega^{(n-2)}] = \{\Phi^{(1)}(X_1, X_2, \omega^{(n-2)})\}$ is finite and includes all the special points of the curve $\Gamma^{(1)}[\omega^{(n-2)}]$, as well as some extra points on $\Gamma^{(1)}[\omega^{(n-2)}]$ which are necessary to ensure the cylindricity of the decomposition. Suppose that by induction we have defined a finite set of the kind $\Omega^{(n-2)}[\omega_n]$, for each value $\omega_n \in I$. The input of the last step of this phase is a curve $\Gamma^{(n-1)} \subset I^n$, obtained by "stretching" along X_n, the set $\Omega^{(n-2)}[\omega_n]$ (which is finite in each X_n-section). An important property of this curve is that (by its actual construction) it is closed under taking projections to certain lower dimensional subspaces. We can then come up with a formula $\Phi^{(n-1)}(X)$, such that the set $\Omega^{(n-1)} = \{\Phi^{(n-1)}(X)\}$ is finite and includes all the special points of the curve $\Gamma^{(n-1)}$. This completes the description of the descending phase. Notice that the set $\{\Phi^{(i)}(X)\}$, $0 \leq i \leq n-1$, has dimension at most $n-i-1$, thus the dimension of $\{\Phi^{(i+1)}(X)\}$ is strictly less than the dimension of $\{\Phi^{(i)}(X)\}$ (at each step the dimension drops - hence the name of this phase). Moreover, for any fixed vector $\omega = (\omega_{i+2}, \dots, \omega_n) \in I^{n-i-1}$ the set $\Omega^{(i+1)}[\omega] = \{\Phi^{(i+1)}(X_1, \dots, X_{i+1}, \omega)\} \subset \mathbb{R}^{i+2}$ is finite and includes all the special points of the curve $\Gamma^{(i)}[\omega] = \{\Phi^{(i)}(X_1, \dots, X_{i+1}, \omega)\}$.

The ascending phase also consists of n steps. In the first step, a cylindrical cell decomposition $\mathcal{D}^{(n-1)}$ of the closed unit interval $L^n_{n-1}(0) = I^n \cap \{X_1 = \cdots = X_{n-1} = 0\}$ compatible with the projection of $Zer(f)$ onto $L^n_{n-1}(0)$ is defined as follows: points in the finite intersection $L^n_{n-1}(0) \cap \Omega^{(n-1)}$, which by construction coincides with the projection of $\Omega^{(n-1)}$ onto $L^n_{n-1}(0)$, are the 0-cylindrical cells and the intervals between them on the X_n-axis are the 1-cylindrical cells. In the next step, we extend $\mathcal{D}^{(n-1)}$ to a cylindrical cell decomposition $\mathcal{D}^{(n-2)}$ of the cube $L^n_{n-2}(0) = I^n \cap \{X_1 = \cdots = X_{n-2} = 0\}$ compatible with the projection of $Zer(f)$ onto this space, as follows. For each cell C in $\mathcal{D}^{(n-1)}$ and any point $p \in C$ consider the finite set $\Omega^{(n-2)}[p] \cap L^n_{n-2}(0)$, which by construction coincides with the projection of $\Omega^{(n-2)}[p]$ onto $L^n_{n-2}(0)$. Points in this intersection and the intervals between them constitute a cylindrical cell decomposition of the closed interval $L^n_{n-2}(0) \cap \{X_n = p\}$. Moreover, the cardinality of $\Omega^{(n-2)}[p] \cap L^n_{n-2}(0)$ is constant over all points $p \in C$. So a cylindrical cell decomposition of the cylinder $Z(C) \cap L^n_{n-2}(0)$ over C, consists of the union, over all points $p \in C$, of the cells in the decomposition of $L^n_{n-2}(0) \cap \{X_n = p\}$. Combining the cell decompositions of $Z(C)$ for all C in $\mathcal{D}^{(n-1)}$, we obtain the desired decomposition $\mathcal{D}^{(n-2)}$ of the 2-dimensional cube $L^n_{n-2}(0)$. Continuing in this way, is possible to define for each i, $0 \leq i \leq n-1$, a cylindrical cell decomposition $\mathcal{D}^{(i)}$ of the cube $L^n_i(0) = I^n \cap \{X_1 = \cdots = X_i = 0\}$ compatible with the projection of $Zer(f)$ onto $L^n_i(0)$. In particular, on the last step of the ascending phase, a cylindrical cell decomposition $\mathcal{D} = \mathcal{D}^{(0)}$ of the cube I^n compatible with $Zer(f)$ is defined.

Section 4.3 is devoted to the actual proof that the cell decomposition sketched above is indeed well defined.

4.1 Reduction to the case of an analytic variety

Let $S \subset G \subset \mathbb{R}^n$ be a $\mathcal{F}$-semianalytic set determined by the quantifier-free $\tilde{\mathcal{L}}_{\mathcal{F}}$-formula

$$\bigvee_{1 \leq i \leq M} \big((f_i = 0) \wedge (g_{i,1} > 0) \wedge \cdots \wedge (g_{i,J_i} > 0)\big).$$

Firstly, we reduce the formula defining the set S to a simple special form which is essentially a single analytic equation. Introduce a new variable X_0 and the function

$$f := \prod_{1 \le i \le M} \Big((f_i^2 + (X_0 - iN)^2) \cdot \prod_{1 \le j \le J_i} (g_{ij}^2 + (X_0 - iN - j)^2) \Big), \tag{4.1}$$

where $N = 1 + \sum_{1 \le i \le M} (1 + J_i)$.

Let $\mathcal{D}$ be a cylindrical cell decomposition of $I^n \times [0, N^2] \subset \mathbb{R}^{n+1}$ compatible with $\{f = 0\} \cap (I^n \times [0, N^2])$, and $\mathcal{D}'$ be the cell decomposition of I^n induced by $\mathcal{D}$.

Obviously $\mathcal{D}'$ is compatible with $\pi(\{f = 0\})$, where $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ is the projection map onto the subspace $\{X_0 = 0\}$ (by the definition of the cylindrical decomposition), but generally, $\pi(\{f = 0\}) \ne S$.

Lemma 4.1.1. *The cylindrical cell decomposition $\mathcal{D}'$ is compatible with S.*

Proof. We need to prove that for any cell C' of $\mathcal{D}'$ either $C' \subset S$ or $C' \cap S = \emptyset$. Suppose that contrary to this, for a cell C' of $\mathcal{D}'$, there are points $x, y \in C'$ such that $x \in \{f_i = 0, g_{i1} > 0, \dots, g_{iJ_i} > 0\}$ for a certain i, and $y \notin S$. In particular, $y \notin \{f_i = 0, g_{i1} > 0, \dots, g_{iJ_i} > 0\}$, i.e. either $g_{ij}(y) \le 0$ for some $1 \le j \le J_i$ or $f_i(y) \ne 0$. Consider first the case $g_{ij}(y) \le 0$, then, since C' is connected, there is a point $z \in C' \cap \{g_{ij} = 0\}$ and therefore a point $(z, iN + j) \in \{f = 0\}$. The point $(z, iN + j)$ belongs to a cell, say C, of $\mathcal{D}$. Note that $\pi(C) = C'$. Clearly, $C \subset \{g_{ij} = 0\} \cap \{X_0 = iN + j\}$. It follows that $C' \subset \{g_{ij} = 0\}$ which contradicts to $x \in C'$ and $g_{ij}(x) > 0$. In the case of $f_i(y) \ne 0$, the point $(y, iN) \notin \{f = 0\}$. The point (y, iN) belongs to a cell, say C'', of $\mathcal{D}$, and $\pi(C'') = C'$. But $C'' \cap \{f = 0\} = \emptyset$, since $\mathcal{D}$ is compatible with $\{f = 0\}$, which is a contradiction, since $x \in C'$ and $f_i(x) = 0$. □

It follows that it is sufficient to construct a cylindrical decomposition of the intersection $\{f = 0\} \cap (I^n \times [0, N^2])$.

4.2 A description of a certain geometric construction

Let G be an open set of $\mathbb{R}^n$ containing the unit cube $I^n = [0, 1]^n$ and $f : G \longrightarrow \mathbb{R}$ an analytic function in n variables from the collection $\mathcal{F}$.

We denote by

$$\rho_l : \quad \mathbb{R}_i^k \longrightarrow \mathbb{R}_i^{k-l+1}, \qquad 1 \le l \le k,\ i \ge 0,$$

the projection map on the coordinate subspace $X_l, X_{l+1} \dots, X_k$.

Let

$$\hat{I}^k := \bigcap_{1 \le j \le k} \{0 < X_j < 1\}, \qquad \tilde{I}^k := I^k \setminus \hat{I}^k,$$

denote the interior and the boundary of the unit cube I^k, respectively.

For any vector $\omega = (\omega_{m+1}, \dots, \omega_k) \in I^{k-m}$, $1 \le m \le k$, and any set $V \subset \mathbb{R}^k$ let

$$V[\omega] := V \cap \{X_{m+1} = \omega_{m+1}, \dots, X_k = \omega_k\};$$

define

- $I_j^k(\alpha)[\omega] := I^k[\omega] \cap \{X_j = \alpha\}$, $1 \le j \le m$, $\alpha \in [0,1]$;
- $I_j^k[\omega] := I_j^k(0)[\omega] \cup I_j^k(1)[\omega]$, $1 \le j \le m$,
  (= union of $(m-1)$-dim. faces of $I^k[\omega]$ which are parallel to $\{X_j = 0\}$);
- $L_0^k[\omega] := I^k[\omega]$ and
  $L_s^k(\alpha)[\omega] := \bigcap_{1 \le j \le s} I_j^k(\alpha)[\omega]$, $1 \le s \le m$, $\alpha \in [0,1]$;
- $L_s^k[\omega] := \bigcap_{1 \le j \le s} I_j^k[\omega]$, $1 \le s \le m$. Observe that $\dim(L_s^k[\omega]) = m - s$.

Let $B_y(t) \subset \mathbb{R}_i^k$ $(i \ge 0)$ denote the ball of radius $t \in \mathbb{R}_i$, centered on y. Define

$$P_k^+(y,t) = B_y(2t) \cap \{X_k = y_k + t\}, \quad P_k^-(y,t) = B_y(2t) \cap \{X_k = y_k - t\}.$$

Definition 4.2.1. *For a $\tilde{\mathcal{R}}_{\mathcal{F}}^{(i)}$-definable curve Γ (a set of dimension at most 1 determined by a $\tilde{\mathcal{L}}_{\mathcal{F}}^{(i)}$-formula) in $\mathbb{R}_i^k$, $i \ge 0$, define:*

1. *$\mathcal{I}(\Gamma)$ to be the set of isolated points of $cl(\Gamma)$*
2. *$\mathcal{E}_k(\Gamma)$ to be the set of all points of local extremum of X_k-coordinate on $cl(\Gamma)$ (so in particular, $\mathcal{I}(\Gamma) \subset \mathcal{E}_k(\Gamma)$);*
3. *$\mathcal{R}_k(\Gamma)$ to be the set of ramification points of $cl(\Gamma)$ with respect to X_k-coordinate (x is a ramification point of $cl(\Gamma)$ with respect to X_k-coordinate, if for all sufficiently small $t \in \mathbb{R}_i$ either $cl(\Gamma) \cap P_k^+(x,t)$ or $cl(\Gamma) \cap P_k^-(x,t)$ consists of at least two distinct points).*
4. *$\mathcal{S}_k(\Gamma) = \mathcal{E}_k(\Gamma) \cup \mathcal{R}_k(\Gamma)$ to be the set of special points of Γ relative to X_k-coordinate.*

Call $\rho_k\big(\mathcal{S}_k(\Gamma)\big)$ the set of special values of the curve Γ relative to X_k-coordinate.

Observe, that it is possible to write down a $\tilde{\mathcal{L}}_{\mathcal{F}}^{(i)}$-formula with quantifiers defining the set $\mathcal{S}_k(\Gamma) \subset \mathbb{R}_i^k$, hence, it has a finite number of $\tilde{\mathcal{R}}_{An}^{(i)}$-definably connected components. In the case when the curve $\Gamma \subset \mathbb{R}_i^k$ is such that for any $\omega_k \in \mathbb{R}_i$, the intersection $\Gamma \cap \{X_k = \omega_k\}$ is finite, then the set $\mathcal{S}_k(\Gamma)$ is also finite.

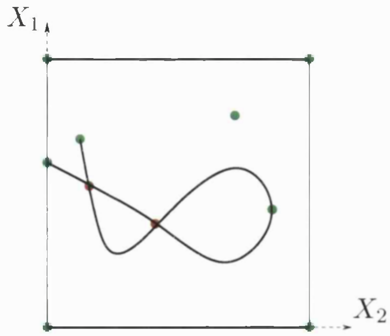


Figure 4-1: The special points of a curve in $\mathbb{R}^2$ relative to X_2-coordinate.

In the remaining part of this section we give a non-constructive description of a certain cylindrical cell decomposition of I^n compatible with $\{f = 0\}$.

Let

$$V = (\{f = 0\} \cap I^n) \cup I_1^n.$$

The description proceeds by induction on n.

Consider the sequence of ordered fields $\mathbb{R} = \mathbb{R}_0 \subset \mathbb{R}_1 \subset \cdots \subset \mathbb{R}_n$ and positive elements with ordering $\varepsilon_0 \gg \varepsilon_1 \gg \cdots \gg \varepsilon_{n-1}$, such that $\varepsilon_i \in \mathbb{R}_{i+1}$ is infinitesimal relative to $\mathbb{R}_i$, $0 \leq i \leq n-1$.

We make several initial steps of the induction.

If $n = 1$, then the analytic set $\{f = 0\}$ is either a finite subset of $\mathbb{R}$ or the whole $\mathbb{R}$. Let

$$\Gamma^{(0)} = V, \qquad \Omega_s^{(0)} = \mathcal{S}_1(\Gamma^{(0)}), \qquad \Omega_0^{(0)} = \Omega_s^{(0)}.$$

For every point $x_1 \in \Omega_s^{(0)}$ consider intersections

$$\Gamma^{(0)} \cap \{X_1 = x_1 + \varepsilon_0\}, \qquad \Gamma^{(0)} \cap \{X_1 = x_1 - \varepsilon_0\},$$

each consisting of at most one point. Denote by $\Omega_1^{(0)}$ the union of these sets. Define

$$\Omega^{(0)} = \Omega_0^{(0)} \cup \Omega_1^{(0)}.$$

Notice that if $Zer(f)$ is finite then $\Omega^{(0)} = \Omega_0^{(0)}$, otherwise $\Omega^{(0)} = \{0, \varepsilon_0, 1 - \varepsilon_0, 1\}$.

Each member of $\Omega_0^{(0)}$ is a zero-dimensional (cylindrical) cell. A cylindrical cell decomposition D of I^1 consists of these points and open intervals on the line, between them. One can enumerate alternatively these points and intervals, by successive non-negative integers j_1, in the increasing along X_1 order, by assigning index $j_1 = 0$ to 0, index $j_1 = 1$ to its neighbouring interval and so on. Observe that $|D| < 2|\Omega^{(0)}|$.

Notice that the set $\Omega_1^{(0)}$ is not needed for the definition of the decomposition D of the closed unit interval. The reason why we actually introduce this set will be explained shortly.

If $n = 2$, then for every fixed value $\omega \in I$ of X_2-coordinate, finite sets $\Omega_0^{(0)}[\omega]$ and $\Omega_1^{(0)}[\omega]$ can be defined by considering $I^2 \cap \{X_2 = \omega\}$ as in case $n = 1$.

Let

$$\widehat{\Gamma}_*^{(1)} := \bigcup_{\omega \in [0,1]} \Omega_*^{(0)}[\omega], \qquad * \in \{0, 1\}.$$

Clearly, $\widehat{\Gamma}_0^{(1)}$ and $\widehat{\Gamma}_1^{(1)}$ are 1-dimensional (not necessarily closed) subsets of $\mathbb{R}_1^2$. Let

$$\widehat{\Gamma}^{(1)} := \widehat{\Gamma}_0^{(1)} \cup \widehat{\Gamma}_1^{(1)}$$

and

$$\Gamma_0^{(1)} := cl(\widehat{\Gamma}_0^{(1)}), \qquad \Gamma_1^{(1)} := cl(\widehat{\Gamma}_1^{(1)}), \qquad \Gamma^{(1)} := cl(\widehat{\Gamma}^{(1)}).$$

Observe that $L_1^2 \subset \widehat{\Gamma}_0^{(1)} \subset \Gamma^{(1)} \subset V$.

Let

$$\Omega_s^{(1)} = \mathcal{S}_2(\Gamma^{(1)}).$$

Denote by $\Omega_0^{(1)}$ the union of finite sets of the kind

$$\widehat{\Gamma}^{(1)} \cap \{X_2 = x_2\}$$

and by $\Omega_1^{(1)}$ the union of finite sets of the kind

$$\widehat{\Gamma}^{(1)} \cap \{X_2 = x_2 + \varepsilon_1\}, \qquad \widehat{\Gamma}^{(1)} \cap \{X_2 = x_2 - \varepsilon_1\},$$

for all points $x_2 \in \rho_2(\Omega_s^{(1)})$.

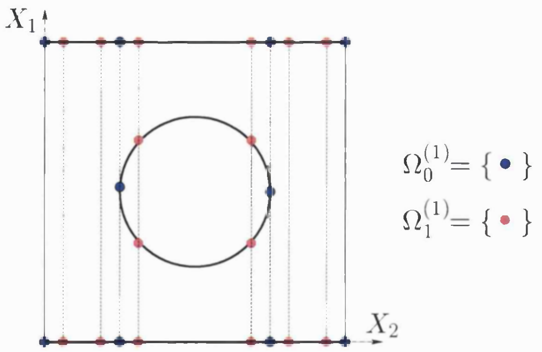


Figure 4-2: Points of the kind $\Omega_0^{(1)}$ and $\Omega_1^{(1)}$ when $Zer(f)$ is a circle in I^2.

Let

$$\Omega^{(1)} = \Omega_0^{(1)} \cup \Omega_1^{(1)}.$$

Notice that in the case when $x = (x_1, x_2) \in \Gamma^{(1)} \setminus \widehat{\Gamma}^{(1)}$, then although, clearly $x \notin \Omega_0^{(1)}$, the point $x_2 \in \rho_2(\Omega_s^{(1)})$. In general, because $L_1^2 \subset \widehat{\Gamma}_0^{(1)}$, projections of points from $\Omega_s^{(1)}$ and $\Omega_0^{(1)}$ on $\{X_1 = 0\}$ coincide, in particular,

$$\rho_2(\Omega_s^{(1)}) = \Omega_0^{(1)} \cap L_1^2(0).$$

Let $y_1 < y_2$ be two neighbouring X_2-coordinates of special points $\Omega_s^{(1)}$ (that is, there is no other special value $y' \in \rho_2(\Omega_s^{(1)})$ such that $y_1 < y' < y_2$). Then for all $y \in (y_1, y_2)$, the set $\Omega_0^{(0)}[y] \subset \{X_2 = y\}$ consists of the same finite number of points. Let us enumerate these points and intervals between them as we did in case $n = 1$, by successive non-negative integers in the increasing along X_1 order.

It is clear that the set of all points having the same index for all $y \in (y_1, y_2)$ is an open interval of the curve $\widehat{\Gamma}^{(1)}$, which is a one-dimensional cylindrical cell being a graph of a continuous function defined on an interval of the 1-dimensional set $L_1^2(0)$. The set of all intervals having the same index for all $y \in (y_1, y_2)$ is an open 2-dimensional cylindrical cell, being the set of points strictly between the non-intersecting graphs of two continuous functions that are defined on an interval in $L_1^2(0)$.

Now we can describe all the cells, zero-, one- and two- dimensional, of the cylindrical decomposition of I^2 that is compatible with V (see Figure 4-3).

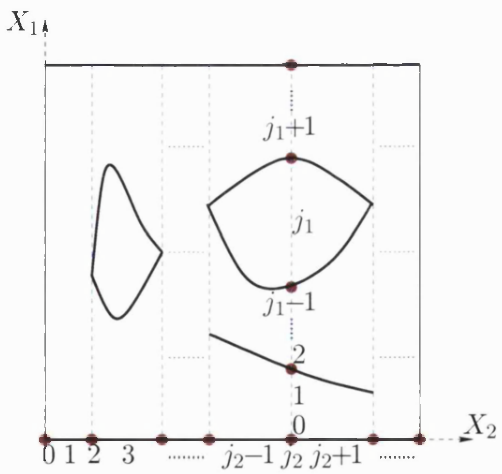


Figure 4-3: Enumeration of cells in a decomposition of I^2

Enumerate each cell by 2-multi-indices of the kind (j_1, j_2) in the following way. Index j_2 numerates (by successive non-negative integers, starting from zero) alternatively points in $\rho_2(\Omega_s^{(1)}) \subset L_1^2(0)$ and intervals between these points on $L_1^2(0)$ (in the increasing along X_2 order). For a fixed value of j_2, index j_1 numerates points in $\Omega_0^{(0)}[y] \subset \{X_2 = y\}$ and the intervals between them (as in case $n = 1$), where y is either the point in $\rho_2(\Omega_s^{(1)})$ having index j_2, or some value in the interval between two neighbouring points of $\rho_2(\Omega_s^{(1)})$ on $L_1^2(0)$ having index j_2.

It is easy to see that the defined family of the cylindrical cells is a cylindrical cell decomposition, say D, of I^2. Cell $C_{i,j}$ having index (i, j) is cylindrical over the cell $C_{0,j}$, with index $(0, j)$, that belongs in the decomposition of $L_1^2(0)$. Notice that when the first index i is odd (because of the way we assign indices) $C_{i,j}$ is a sector over $C_{0,j}$, otherwise is a section.

For example when V is a circle in I^2 (see Figure 4-2) we obtain a decomposition with 10 zero dimensional cells, 14 one dimensional cells and 5 two dimensional cells.

Observe that the number of cells in this decomposition is at most $4|\Omega^{(1)}|$.

We now explain why it is essential to introduce the set $\Omega_1^{(0)}[y]$ for each $y \in [0, 1]$. Consider for example, the set $V = \left(I_1^2 \cup \{X_1 = 1/2\} \cup \{X_2 = 1/2\}\right) \cap I^2$ together with its corresponding curves $\Gamma_0^{(1)}$ and $\Gamma_1^{(1)}$, shown in Figure 4-4.

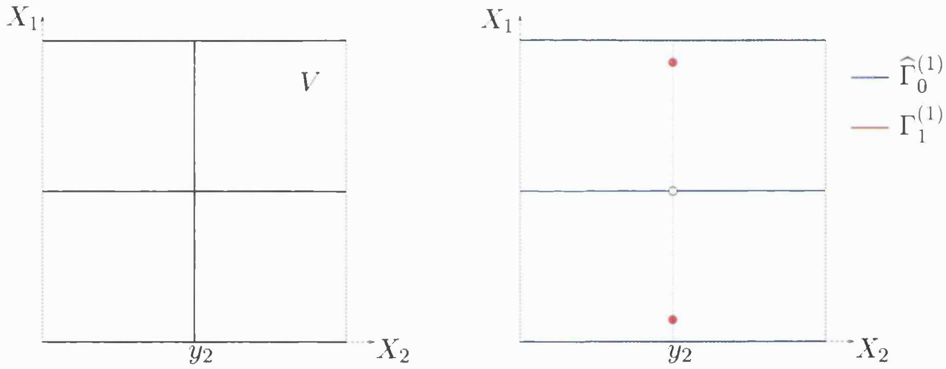


Figure 4-4: An example where the curve $\Gamma_1^{(1)}$ is essential.

Clealry, $y_2 = 1/2$ is a special value of $\Gamma_1^{(1)}$ relative to X_2. Observe that by looking only at the set of special points of the curve $\Gamma_0^{(1)}$ relative to X_2, we have no way to tell that

$$I^2 \cap \{X_2 = 1/2\} \subset V$$

but

$$I^2 \cap \{X_2 = 1/2\} \not\subset \{X_2 = 1/2\} \cap cl\left(V \cap \{X_2 \neq 1/2\}\right).$$

Let x_1, y_1, with $x_1 < y_1$, be any two neighbouring points from $\Omega_0^{(0)}[\omega_2]$. By the definition of this set, if for any point $\omega_1 \in (x_1, y_1)$ and some $\omega_2 \in [0, 1]$ the intersection $V \cap \{X_2 = \omega_2\} \cap \{X_1 = \omega_1\}$ is nonempty, then the interval $(x_1, y_1) \subset V \cap \{X_2 = \omega_2\}$. Taking

$\varepsilon_0 > 0$ to be infinitesimal relative to $\mathbb{R}$ in the definitinion of $\Omega_1^{(0)}[\omega_2]$, makes sure, for example, that $x_1 < x_1 + \varepsilon_0 < y_1$ and moreover, if $\Omega_1^{(0)}[\omega_2] \cap \{x_1 < X_1 < y_1\} \neq \emptyset$ then $(x_1 + \varepsilon_0, \omega_2)$ lies on $\Gamma_1^{(1)}$ but not on the curve $\Gamma_0^{(1)}$ which is defined over $\mathbb{R}$. So any isolated points of $\Gamma_1^{(1)}$ are necessarily also isolated points of $\Gamma^{(1)}$.

We proceed to the description of a general induction step i, $0 \leq i \leq n-2$.

Suppose that for every value $\omega_{i+2} \in I$ and every vector $\omega = (\omega_{i+3}, \dots, \omega_n) \in I^{n-i-2}$, a finite set of points of the kind $\Omega_*^{(i)}[\omega_{i+2}, \omega]$, $* \in \{0,1\}$, can be defined by applying the inductive hypothesis to $I^n[\omega_{i+2}, \dots, \omega_n] \subset \mathbb{R}_{i+1}^{i+1}$.

An important property of this set, is that there is a $\tilde{\mathcal{L}}_{\mathcal{F}}^{(i+1)}$-formula with quantifiers, say $\Phi_*^{(i)}(X_1, \dots, X_{n-1}, X_n)$, having free variables $X_1, \dots, X_n$ and not depending on ω, such that the replacement of $X_{i+2}, \dots, X_n$ by $\omega_{i+2}, \dots, \omega_n$ respectively, gives a formula $\Phi_*^{(n-2)}(X_1, \dots, X_{i+1}, \omega_{i+2}, \omega)$ in free variables $X_1, \dots, X_{i+1}$ defining the set $\Omega_*^{(i)}[\omega_{i+2}, \omega] \subset \mathbb{R}_{i+1}^{i+1}$. Let

$$\begin{aligned} \widehat{\Gamma}_*^{(i+1)}[\omega] &:= \{\Phi_*^{(i)}(X_1, \dots, X_{i+2}, \omega)\} \\ &= \bigcup_{\omega_{i+2} \in [0,1]} \Omega_*^{(i)}[\omega_{i+2}, \omega]. \end{aligned}$$

Clearly, $\widehat{\Gamma}_*^{(i+1)}[\omega]$ is a 1-dimensional (not necessarily closed) subset of $\mathbb{R}_{i+1}^{i+2}$. Let

$$\widehat{\Gamma}^{(i+1)}[\omega] = \widehat{\Gamma}_0^{(i+1)}[\omega] \cup \widehat{\Gamma}_1^{(i+1)}[\omega].$$

Define

$$\Gamma_0^{(i+1)}[\omega] := cl\big(\widehat{\Gamma}_0^{(i+1)}[\omega]\big), \qquad \Gamma_1^{(i+1)}[\omega] := cl\big(\widehat{\Gamma}_1^{(i+1)}[\omega]\big)$$

and

$$\Gamma^{(i+1)}[\omega] := cl\big(\widehat{\Gamma}^{(i+1)}[\omega]\big).$$

Observe that

$$L_{i+1}^n[\omega] \subset \widehat{\Gamma}_0^{(i+1)}[\omega] \subset \Gamma^{(i+1)}[\omega] \subset V.$$

Moreover, for any $k = 2, \dots, i+1$ and $* \in \{0,1\}$ by the definition of $\Gamma_*^{(i+1)}[\omega]$, we have the inclusions

$$\rho_k\big(\widehat{\Gamma}_*^{(i+1)}[\omega]\big) \subset \widehat{\Gamma}_*^{(i+1)}[\omega], \qquad \rho_k\big(\Gamma_*^{(i+1)}[\omega]\big) \subset \Gamma_*^{(i+1)}[\omega],$$

where ρ_k denotes the projection on coordinate subspace $X_k, X_{k+1} \dots, X_n$.

Applying the inductive hypothesis to $I^n[\omega_{i+2}, \dots, \omega_n]$, using the set $\Omega_0^{(i)}[\omega_{i+2}, \omega]$ we obtain a cylindrical cell decomposition of this section.

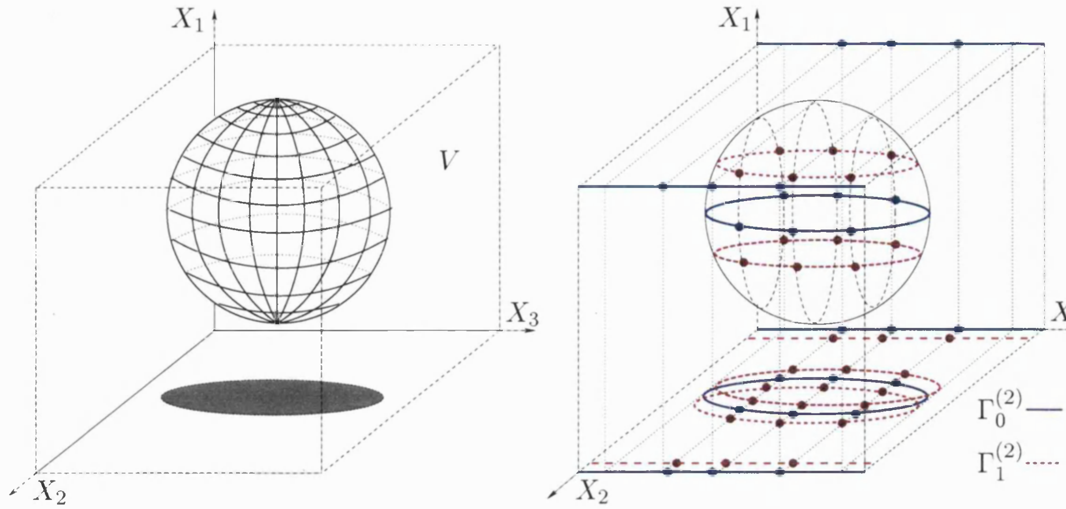


Figure 4-5: Let $Zer(f)$ be a sphere in I^3. First consider points of the kind $\Omega^{(1)}$ for each X_3-section (as in case $n = 2$ for a circle – see Figure 4-2) and we then "stretch" them along X_3 to obtain the curve $\Gamma^{(2)}$ (for clarity, the curve $\Gamma^{(2)} \cap \{X_1 = 1\}$ is omitted).

Let

$$\Omega_s^{(i+1)}[\omega] := \mathcal{S}_n(\Gamma^{(i+1)}[\omega]).$$

The set $\rho_{i+2}(\Omega_s^{(i+1)}[\omega])$ of special values of $\Gamma^{(i+1)}[\omega]$ relative to X_{i+2}-coordinate includes all values of X_{i+2}-coordinate for which there is a change in the number of points in the finite intersection of the curve $\Gamma^{(i+1)}[\omega]$ with a hyperplane orthogonal to X_{i+2}-axis, as it sweeps along X_{i+2}.

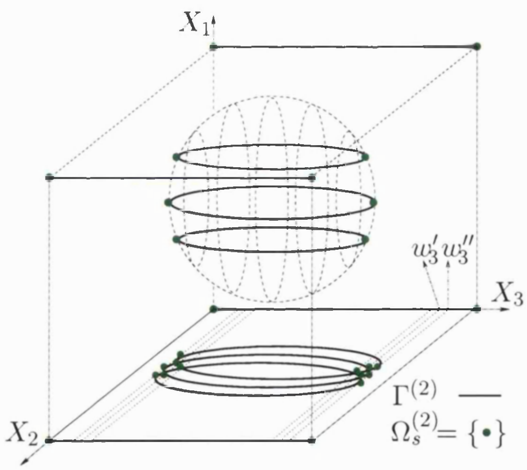


Figure 4-6: The special points $\Omega_s^{(2)}$ of the curve $\Gamma^{(2)}$ when $Zer(f)$ is a sphere in I^3.

Denote by $\Omega_0^{(i+1)}[\omega]$ the union of finite sets of the kind

$$\widehat{\Gamma}^{(i+1)}[\omega] \cap \{X_{i+2} = x_{i+2}\}$$

and by $\Omega_1^{(i+1)}[\omega]$ the union of finite sets of the kind

$$\widehat{\Gamma}^{(i+1)}[\omega] \cap \{X_{i+2} = x_{i+2} + \varepsilon_{i+1}\}, \qquad \widehat{\Gamma}^{(i+1)}[\omega] \cap \{X_{i+2} = x_{i+2} - \varepsilon_{i+1}\},$$

for all points $x_{i+2} \in \rho_{i+2}\big(\Omega_s^{(i+1)}[\omega]\big)$.

Let

$$\Omega^{(i+1)}[\omega] = \Omega_0^{(i+1)}[\omega] \cup \Omega_1^{(i+1)}[\omega].$$

Notice that if $x \in \Gamma^{(i+1)}[\omega] \setminus \widehat{\Gamma}^{(i+1)}[\omega]$, then although $x \notin \Omega_0^{(i+1)}[\omega]$, the point $x_{i+2} \in \rho_{i+2}\big(\Omega_0^{(i+1)}[\omega]\big)$. In general, since $L_{i+1}^n(0)[\omega] \subset \widehat{\Gamma}_0^{(i+1)}[\omega]$, it follows that

$$\rho_{i+2}\big(\Omega_s^{(i+1)}[\omega]\big) = \rho_{i+2}\big(\Omega_0^{(i+1)}[\omega]\big) = \Omega_0^{(i+1)}[\omega] \cap L_{i+1}^n(0)[\omega].$$

Let the index j_{i+2} (non-negative integer, taking 0 as its first value) enumerate successively, in the increasing along X_{i+2} order, alternatively, points in $\Omega_0^{(i+1)}[\omega] \cap L_{i+1}^n(0)[\omega]$ and intervals between these points on $L_{i+1}^n(0)[\omega]$.

Let $y_1 < y_2$ be two neighbouring points from $\rho_{i+2}\big(\Omega_s^{(i+1)}[\omega]\big)$. Assume that the interval (y_1, y_2) is indexed by j_{i+2}. It follows from inductive hypothesis that there is a certain cylindrical cell decomposition of the intersection $I^n[\omega_{i+2}, \omega]$ compatible with $V[\omega_{i+2}, \omega]$ in which all cells are enumerated by $(i+1)$-multi-indices. In the next section we shall prove that for all $y \in (y_1, y_2)$ the sets of multi-indices in $I^n[y, \omega]$ coincide, moreover a fixed multi-index corresponds to cells of the same dimension and finally, the union of all p-cells ($p = 0, 1, \dots, i+1$) having the same multi-index (say, $(j_1, \dots, j_{i+1})$) for all $y \in (y_1, y_2)$ is a cylindrical $(p+1)$-cell of a cell decomposition of $I^n[\omega]$ to which we shall assign multi-index $(j_1, \dots, j_{i+1}, j_{i+2})$.

Let y be the point in $\rho_{i+2}\big(\Omega_s^{(i+1)}[\omega]\big)$ having the index j_{i+2}. By the inductive hypothesis there is a cylindrical cell decomposition of $I^n[y, \omega]$. The cells of this decomposition are also the elements of a cell decomposition of $I^n[\omega]$. If a cell in $I^n[y, \omega]$ has a multi-index $(j_1, \dots, j_{i+1})$, then considering it as a cell in $I^n[\omega]$ we assign to it the multi-index $(j_1, \dots, j_{i+1}, j_{i+2})$.

Notice that if a cell has index $(j_1, \dots, j_{i+2})$, then its dimension is equal to

$$(j_1 \, mod \, 2) + \cdots + (j_{i+2} \, mod \, 2).$$

Observe that the total number of cells in the described decomposition of $I^n[\omega]$ is at most $2^{i+2}|\Omega^{(i+1)}[\omega]|$.

This finishes the description of the general induction step.

It seems a good idea, at this point, to try and explain briefly why in the definition of the set $\Omega_1^{(k)}[\omega]$ given above, for $0 \le k \le n-1$ and some $\omega = (\omega_{k+2}, \dots, \omega_n) \in I^{n-k-1}$, we had to use a positive element $\varepsilon_k \in \mathbb{R}_{k+1}$ infinitesimal relative to the field $\mathbb{R}_k$ over which the curve $\widehat{\Gamma}^{(k)}[\omega]$ is defined. For any two neighbouring X_{k+1}-coordinates x_{k+1}, y_{k+1} (with $x_{k+1} < y_{k+1}$) of points from $\Omega_0^{(k)}[\omega]$, we need to ensure that for some value z_{k+1} strictly between x_{k+1} and y_{k+1}, points in the intersection $\widehat{\Gamma}^{(k)}[\omega] \cap \{X_{k+1} = z_{k+1}\}$ are included in $\Omega_1^{(k)}[\omega]$. One possible approach which turns out to lead to (comparatively) simpler formulas in the construction stage that follows in Chapter 5, is to take $z_{k+1} = x_{k+1} + d$, where d is a sufficiently small positive element of $\mathbb{R}_k$ (another possibility is to take z_{k+1} to be the midpoint of the values x_{k+1} and y_{k+1}, this is explored in Appendix B). But unfortunately, there is no way to determine in advance a lower bound on the least distance between different values of X_{k+1}-coordinates of points from the set $\Omega_s^{(k)}[\omega]$ (which is definable by some first order formula involving real analytic functions).

The reason for introducing sets of the kind $\Omega_1^{(k)}[\omega]$ is to provide additional information regarding the geometrical structure of X_{k+1}-sections of $V[\omega]$, strictly between every consecutive pair of special values of $\Gamma^{(k)}[\omega]$ relative to X_{k+1}-coordinate. This sort of information may play a crucial role in the process of identifying, during some later step $k+l$, $l \ge 1$ of the inductive description, all the "critical" values of the coordinate function X_{k+l+1} that are essential for the construction of a cylindrical cell decomposition of I^n compatible with V. For example it is possible that the curve $\Gamma^{(k+l)}[\omega'] \subset V[\omega'] \subset I^n[\omega']$, where $\omega' = (\omega_{k+l+2}, \dots, \omega_n)$ with $0 \le \omega_j \le 1$, $k+l+2 \le j \le n$, has a special value y_{k+l+1} relative to X_{k+l+1}-coordinate, such that

1. $V[\omega'] \cap \{X_{k+l+1} = y_{k+l+1}\} \not\subset \{X_{k+l+1} = y_{k+l+1}\} \cap cl\big(V[\omega'] \cap \{X_{k+l+1} \neq y_{k+l+1}\}\big)$;
2. if $y = (y_1, \dots, y_{k+l+1}) \in \mathcal{S}_{k+l+1}\big(\Gamma^{(k+l)}[\omega']\big) \cap \{X_{k+l+1} = y_{k+l+1}\}$, then there exist points $y^{(j)} = (y_1^{(j)}, \dots, y_j^{(j)})$, $k+1 \le j \le k+l+1$, such that

$$y^{(k+l+1)} \in \Gamma^{(k+l)}[\omega'] \cap \{X_{k+l+1} = y_{k+l+1}\},$$

$$y^{(i)} \in \Gamma^{(i-1)}[y_{i+1}^{(i+2)}, \cdots, y_{k+l}^{(k+l+1)}, y_{k+l+1}, \omega'] \cap \{X_i = y_i^{(i+1)}\},\ i = k+l, \dots, k+2,$$

$$y^{(k+1)} \in \Omega_1^{(k)}[y_{k+1}^{(k+2)}, y_{k+2}^{(k+3)}, \cdots, y_{k+l}^{(k+l+1)}, y_{k+l+1}^{(k+l+1)}, \omega'].$$

What this is saying, is that, either y is itself a point in

$$\{X_{k+2} = y_{k+2}, \dots, X_{k+l+1} = y_{k+l+1}\} \cap \Omega_1^{(k)}[y_{k+2}, \dots, y_{k+l+1}, \omega'],$$

or y is actually "derived" by points from sets of the kind $\Omega_1^{(k)}$.

This means, that in the absence of the sets of the kind $\Omega_1^{(k)}$, the value y_{k+l+1} of the X_{k+l+1}-coordinate, which is indeed necessary for the construction of any cylindrical cell decomposition of I^n compatible with V, would not have been detected. This is demostrated in the examples shown in Figure 4-7 (for $k = 0$, $l = 1, 2$) and Figure 4-8 (for $k = 1$, $l = 1$).

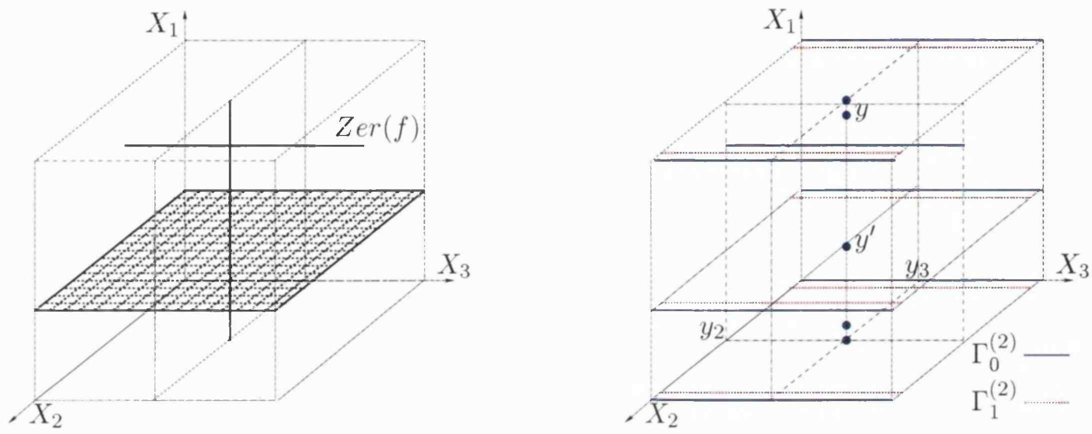


Figure 4-7: The set $\Omega_1^{(0)}[y_2, y_3]$ is essential for the description of a cylindrical cell decomposition of I^3 compatible with $Zer(f)$. It gives rise to isolated points of $\Gamma^{(1)}[y_3]$ as well as to isolated points of $\Gamma^{(2)}$ (for example, the point $y \in \Omega_1^{(0)}[y_2, y_3]$, while $y' \in \Gamma^{(1)}[y_3] \cap \{X_2 = y_2\}$).

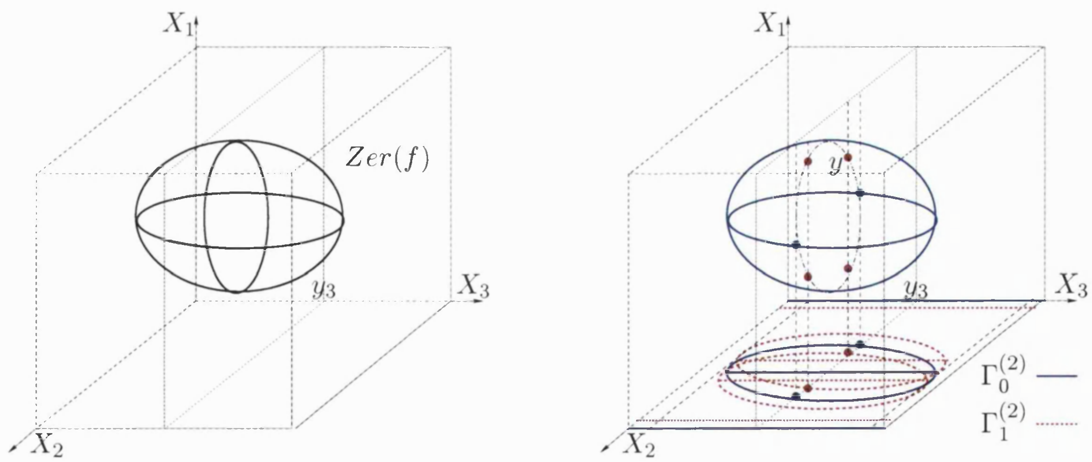


Figure 4-8: The set $\Omega_1^{(1)}[y_3]$ is essential for the description of a cylindrical cell decomposition of I^3 compatible with $Zer(f)$. It gives rise to the only special points of the curve $\Gamma^{(2)}$ relative to X_3-coordinate, that lie in the hyperplane $\{X_3 = y_3\}$.

4.3 Establishing the correctness of the cell decomposition description

In what follows we prove some propositions which where used in the previous section for the description of the cylindrical cell decomposition.

Let $w' = (w'_1, \dots, w'_n)$, $w'' = (w''_1, \dots, w''_n)$, with $w'_n < w''_n$, be two neighbouring special points of $\Gamma^{(n-1)}$ relative to X_n-coordinate, i.e., $w', w'' \in \Omega_s^{(n-1)}$ and

$$\Omega_s^{(n-1)} \cap \{w'_n < X_n < w''_n\} = \emptyset,$$

and let the values z_1, z_2, with $z_1 < z_2$, belong in the interval (w'_n, w''_n).

According to the inductive hypothesis (of the induction described in previous section) on both $I^n[z_1]$ and $I^n[z_2]$ certain cylindrical cell decompositions are defined. Let $(i_1, \dots, i_{n-1})$ be a multi-index of a cylindrical p-cell C_1 in $I^n[z_1]$.

Lemma 4.3.1. *There exists a cylindrical p-cell C_2 in $I^n[z_2]$ having the same multi-index $(i_1, \dots, i_{n-1})$ such that $C_1 \subset V$ if and only if $C_2 \subset V$.*

Proof. It follows from the definition of values w'_n and w''_n that the curve $\widehat{\Gamma}^{(n-1)} \cap \{z_1 \leq X_n \leq z_2\}$ consists of a finite number of disjoint curve segments each of which is homeomorphic to $[0,1]$ and on each of which X_n monotonically increases. In particular, this is true for both $\widehat{\Gamma}_0^{(n-1)} \cap \{z_1 \leq X_n \leq z_2\}$ and $\widehat{\Gamma}_1^{(n-1)} \cap \{z_1 \leq X_n \leq z_2\}$. Therefore, the sets of points

$$\Omega_*^{(n-2)}[z_1] \subset I^n[z_1], \qquad \Omega_*^{(n-2)}[z_2] \subset I^n[z_2], \qquad * \in \{0,1\},$$

are in a natural bijective correspondence: $x \in \Omega_*^{(n-2)}[z_1]$ corresponds to $y \in \Omega_*^{(n-2)}[z_2]$ if and only if x and y belong to the same connected component of the intersection $\widehat{\Gamma}_*^{(n-1)} \cap \{z_1 \leq X_n \leq z_2\}$.

Let the points

$$x^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)}), \quad x^{(2)} = (x_1^{(2)}, \dots, x_n^{(2)})$$

belong to $\Omega_0^{(n-2)}[z_1]$ and j, $1 \leq j \leq n-1$, be an index such that

1. $x_n^{(1)} = x_n^{(2)} = z_1$, $x_{n-1}^{(1)} = x_{n-1}^{(2)}, \dots, x_{j+1}^{(1)} = x_{j+1}^{(2)}$ and $x_j^{(1)} < x_j^{(2)}$;

2. in the interval $(x_j^{(1)}, x_j^{(2)})$ there is no j-coordinate of any point from

$$\Omega_0^{(n-2)}[z_1] \cap \{X_{n-1} = x_{n-1}^{(1)}, \dots, X_{j+1} = x_{j+1}^{(1)}\}.$$

Then there exist certain points

$$y^{(1)} = (y_1^{(1)}, \dots, y_n^{(1)}),\ y^{(2)} = (y_1^{(2)}, \dots, y_n^{(2)}) \in \Omega_0^{(n-2)}[z_2]$$

which are the images of $x^{(1)}, x^{(2)}$ respectively, under the bijective correspondence described above.

To prove the lemma, it is sufficient to show that the points $y^{(1)}, y^{(2)}$ satisfy the conditions similar to (1), (2) for $x^{(1)}, x^{(2)}$. Namely:

(i) $y_n^{(1)} = y_n^{(2)} = z_2,\ y_{n-1}^{(1)} = y_{n-1}^{(2)}, \dots, y_{j+1}^{(1)} = y_{j+1}^{(2)}$ and $y_j^{(1)} < y_j^{(2)}$;

(ii) in the interval $(y_j^{(1)}, y_j^{(2)})$ there is no j-coordinate of any point from the intersection

$$\Omega_0^{(n-2)}[z_2] \cap \{X_{n-1} = y_{n-1}^{(1)}, \dots, X_{j+1} = y_{j+1}^{(1)}\}.$$

Indeed, suppose that (i), (ii) are established. Index i_{n-1}, by definition, either numerates the X_{n-1}-projection of a special point (rel. to X_{n-1}) of the curve $\Gamma^{(n-2)}[z_1] \subset I^n[z_1]$, say, $x^{(1)} \in \Omega_s^{(n-2)}[z_1]$, or an interval on $L_{n-2}^n[0][z_1]$ between two neighbouring X_{n-1}-projections of special points (rel. to X_{n-1}) of $\Gamma^{(n-2)}[z_1]$, say, $x^{(1)}, x^{(t)} \in \Omega_s^{(n-2)}[z_1]$, for some $t > 1$. Note that $x^{(1)}, x^{(t)} \in \Omega_0^{(n-2)}[z_1] \subset \Gamma_0^{(n-1)}$.

In the first case, according to the properties (i), (ii), the image point $y^{(1)} \in \Gamma_0^{(n-2)}[z_2] \subset I^n[z_2]$ of $x^{(1)}$ (under the described bijective correspondence) has the same last index i_{n-1}. In the second case, by the same argument, the interval on X_{n-1}-axis between X_{n-1}-coordinates of the image points $y^{(1)}, y^{(t)}$ has number i_{n-1}. We now show that this implies the existence of a cell C_2 in $I^n[z_2]$ with the last component of the multi-index equal to i_{n-1} such that $C_2 \subset V$ if and only if $C_1 \subset V$. By inductive hypothesis $C_j \cap V = \emptyset$ or $C_j \subset V$ for $j = 1, 2$.

Indeed, in the first case above, the intersection $\Gamma^{(n-2)}[z_2] \cap \{X_{n-1} = y_{n-1}^{(1)}\}$ contains $y^{(1)}$ and therefore, there exists a cell C_2 such that $C_2 \cap V \cap I^n[z_2]$ is non-empty.

We now examine the second case.

Consider the plane $L_{n-2}^n[0]$ equipped with coordinates X_{n-1}, X_n. Observe that the curve $\Gamma_1^{(n-1)} \cap L_{n-2}^n[0]$ is obtained as the ε_{n-2}-shift of $\Gamma_0^{(n-1)} \cap L_{n-2}^n[0]$ along the X_{n-1}-coordinate. As a result the distance $|x_{n-1}^{(t)} - x_{n-1}^{(1)}| \neq \varepsilon_{n-2}$.

Suppose that $|x_{n-1}^{(t)} - x_{n-1}^{(1)}| < \varepsilon_{n-2}$. Since $\Gamma_0^{(n-1)}$ is defined over the field $\mathbb{R}_{n-2}$ not including ε_{n-2}, this can only be possible if

- either w' or w'' belong, either in the intersection $\Gamma_0^{(n-1)} \cap \Gamma_1^{(n-1)}$, or in the set $\mathcal{R}_n(\Gamma_1^{(n-1)})$ and so in particular w' or w'' is a ramification point of $\Gamma^{(n-1)}$, and

- there exists $w = (w_1, \ldots, w_n) \in \mathcal{R}_n(\Gamma_0^{(n-1)} \cap L_{n-2}^n[0])$, such that $|w_n' - w_n| = O(\varepsilon_{n-2})$ and $|w_n'' - w_n| = O(\varepsilon_{n-2})$, implying that $(w_n'' - w_n')$ is of the same order.

It follows that $|y_{n-1}^{(t)} - y_{n-1}^{(1)}| < \varepsilon_{n-2}$. Similarly if $|x_{n-1}^{(t)} - x_{n-1}^{(1)}| > \varepsilon_{n-2}$ then the distance $|y_{n-1}^{(t)} - y_{n-1}^{(1)}| > \varepsilon_{n-2}$.

Let $\Gamma_{x^{(1)},y^{(1)}}$ (respectively, $\Gamma_{x^{(t)},y^{(t)}}$) be the connected component of the intersection $\widehat{\Gamma}_0^{(n-1)} \cap \{z_1 \leq X_n \leq z_2\}$ realizing the bijection between $x^{(1)}, y^{(1)}$ (respectively, between $x^{(t)}, y^{(t)}$).

By inductive hypothesis applied to the section $I^n[z]$, $z \in \{w_n' < X_n < w_n''\}$, for any two values p_1, p_2 of X_{n-1}-coordinate lying strictly between the X_{n-1}-coordinates of the points $\Gamma_{x^{(1)},y^{(1)}} \cap \{X_n = z\}$ and $\Gamma_{x^{(t)},y^{(t)}} \cap \{X_n = z\}$, certain cylindrical cell decompositions can be defined on both $I^n[z] \cap \{X_{n-1} = p_1\}$ and $I^n[z] \cap \{X_{n-1} = p_2\}$, so that there exists a cell having index $(i_1, \ldots, i_{n-2})$ in one of the sections if and only if there exists a cell having the same index in the other section.

Define μ_z to be the mean value of the X_{n-1}-coordinates of the points

$$\Gamma_{x^{(1)},y^{(1)}} \cap \{X_n = z\} \qquad \& \qquad \Gamma_{x^{(t)},y^{(t)}} \cap \{X_n = z\}.$$

Let

$$\Omega_m^{(n-2)}[z] = \widehat{\Gamma}_0^{(n-2)}[z] \cap \{X_{n-1} = \mu_z\}$$

and

$$\Gamma_m^{(n-1)} = \bigcup_{z \in \{z_1 \leq X_n \leq z_2\}} \Omega_m^{(n-2)}[z].$$

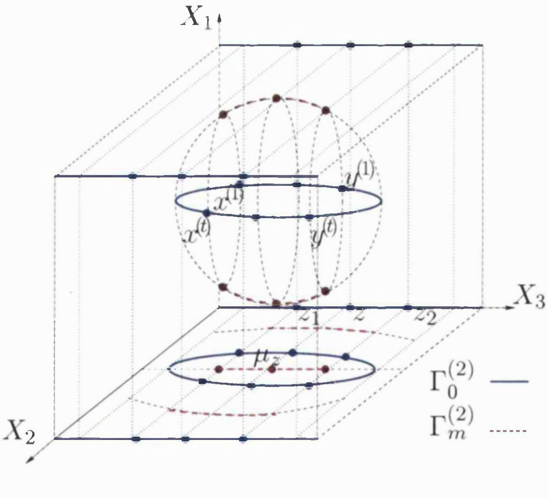


Figure 4-9: The curve $\Gamma_m^{(2)}$ when $Zer(f)$ is a sphere in I^3.

Claim 4.3.2. *The curve $\Gamma_m^{(n-1)} \cap \{z_1 \leq X_n \leq z_2\}$ consists of a finite number of disjoint curve segments each of which is homeomorphic to $[0,1]$ and on each of which X_n monotonically increases.*

Proof of the Claim. Contrary to this claim, assume that $\Gamma_m^{(n-1)} \cap \{z_1 \leq X_n \leq z_2\}$ does not consist of a finite number of disjoint curve segments as described above. One of the following must be true: either

A. $|x_{n-1}^{(t)} - x_{n-1}^{(1)}| > \varepsilon_{n-2}$ and $|y_{n-1}^{(t)} - y_{n-1}^{(1)}| > \varepsilon_{n-2}$ or

B. $|x_{n-1}^{(t)} - x_{n-1}^{(1)}| < \varepsilon_{n-2}$ and $|y_{n-1}^{(t)} - y_{n-1}^{(1)}| < \varepsilon_{n-2}$.

If A is true, then the above assumption implies in particular, that similarly, the curve $\Gamma_1^{(n-1)} \cap \{z_1 \leq X_n \leq z_2\}$ does not consist of a finite number of disjoint curve segments, with each one homeomorphic to $[0,1]$, and on which X_n monotonically increases. This contradicts the fact that there are no special points of $\Gamma_1^{(n-1)}$ (rel. to X_n) with X_n-coordinate in the interval (w'_n, w''_n).

If B is true, then the curve $\Gamma_m^{(n-1)}$, which is defined over the field $\mathbb{R}_{n-2}$ not including ε_{n-2}, must have a special point (rel. to X_n), say w''', such that its X_n-coordinate w'''_n lies strictly between w'_n and w''_n, where $w''_n - w'_n = O(\varepsilon_{n-2})$. Due to the construction of the curve $\Gamma^{(n-1)}$, there must be some special value (rel. to X_n), say w_n, of the curve $\Gamma_0^{(n-1)}$ (which is also defined over $\mathbb{R}_{n-2}$), that is $O(\varepsilon_{n-2})$-close to both w'_n and w''_n (e.g, in the example of Figure 4-6 take $w_n = w''_n$). We obtain a contradiction, since this implies that $w'''_n - w_n = O(\varepsilon_{n-2})$.

This finishes the proof of the claim.

As a result, there exists a point, say

$$x' = (x'_1, \ldots, x'_n) \in \Gamma^{(n-2)}[z_1] \cap \{X_{n-1} = (x_{n-1}^{(1)} + x_{n-1}^{(t)})/2\},$$

which by definition belongs to $\Omega_m^{(n-2)}[z_1]$, and a connected component Γ' of $\Gamma_m^{(n-1)} \cap \{z_1 \leq X_n \leq z_2\}$ containing x'. Due to (i), (ii), the intersection

$$(\Gamma' \cap I^n[z_2]) \subset \Gamma^{(n-2)}[z_2] \cap \{y_{n-1}^{(1)} \leq X_{n-1} \leq y_{n-1}^{(t)}\},$$

and thus

$$\Gamma^{(n-2)}[z_2] \cap \{y_{n-1}^{(1)} < X_{n-1} < y_{n-1}^{(t)}\} \neq \emptyset,$$

i.e., there exists a non-empty cell C_2 in $V \cap I^n[z_2]$ with the last component of the multi-index equal to i_{n-1}.

Repeating this argument by induction for indices $i_{n-2}, \ldots, i_1$ we prove that there exists a p-cell on $I^n[z_2]$ with multi-index $(i_1, \ldots, i_{n-1})$.

Now we proceed to the proof of conditions (i), (ii).

Suppose that (i) is false because there exists $k \in \{j+1, \dots, n-1\}$ such that $y_k^{(1)} \neq y_k^{(2)}$. Let $s \in \{j+1, \dots, n-1\}$ be the maximum among such numbers k and $y_s^{(1)} < y_s^{(2)}$.

According to its actual definition, the curve $\Gamma_0^{(n-1)}$ is closed under images of the map ρ_s, where ρ_s denotes the projection onto the subspace $\{X_1 = \cdots = X_{s-1} = 0\}$ equipped with coordinates $X_s, X_{s+1} \dots, X_n$.

In particular, $\rho_s(\Omega_0^{(n-2)}[z]) \subset \Omega_0^{(n-2)}[z] \subset I^n[z]$, where $z = z_1$ or z_2.

Define the points $y^{(3)} = \rho_s(y^{(1)}), y^{(4)} = \rho_s(y^{(2)})$ which belong in

$$\Omega_0^{(n-2)}[z_2] \cap \{X_{s+1} = y_{s+1}^{(1)}, \dots, X_n = y_n^{(1)}\}$$

and the point $x^{(3)} = \rho_s(x^{(1)})$, which belong in

$$\Omega_0^{(n-2)}[z_1] \cap \{X_{s+1} = x_{s+1}^{(1)}, \dots, X_n = x_n^{(1)}\}.$$

Thus $y_i^{(3)} = y_i^{(4)} = x_i^{(3)} = 0$ for every $1 \leq i < s$, $y_s^{(3)} = y_s^{(1)}$, $y_s^{(4)} = y_s^{(2)}$ (so $y_s^{(3)} < y_s^{(4)}$) and $x_s^{(3)} = x_s^{(1)} = x_s^{(2)}$, since by assumption $\rho_s(x^{(1)}) = \rho_s(x^{(2)})$.

Define the curves $\Gamma' = \rho_s(\Gamma_{x^{(1)},y^{(1)}})$ and $\Gamma'' = \rho_s(\Gamma_{x^{(2)},y^{(2)}})$. Both projections are definably connected and contained in $\Gamma_0^{(n-1)}$.

Then $y^{(3)} \neq y^{(4)}$, $y^{(3)} \in \Gamma'$, $y^{(4)} \in \Gamma''$, and $x^{(3)} \in \Gamma' \cap \Gamma''$. It follows that $x^{(3)}$ is a ramification (and thus a special) point of $\Gamma_0^{(n-1)}$ (rel. to X_n), and so $x^{(3)} \in \Omega_s^{(n-1)}$, which contradicts the choice of z_1, z_2.

This finishes the proof of the fact that for all $k \in \{j+1, \dots, n\}$ the equality $y_k^{(1)} = y_k^{(2)}$ is true.

The inequality $y_j^{(1)} < y_j^{(2)}$ (see (i)) can be proved by a symmetric argument. The property (ii) can be proved similarly. □

For a value z of X_n-coordinate such that $w_n' < z < w_n''$ denote by $C_z^{(i_1,\dots,i_{n-1})}$ the cylindrical p-dimensional cell in $I^n[z]$ having the multi-index $(i_1, \dots, i_{n-1})$.

Lemma 4.3.3. *The set*

$$C := C^{(i_1,\dots,i_{n-1})} := \bigcup_{z \in (w_n', w_n'')} C_z^{(i_1,\dots,i_{n-1})}$$

is a cylindrical $(p+1)$-cell in I^n.

Proof. We shall prove by induction on n and p that C is a homeomorphic image of an open $(p+1)$-dimensional cube $(0,1)^{p+1}$.

Define the map

$$F : (0,1)^{p+1} \longrightarrow C$$

in the following way.

Among components of the multi-index $(i_1, \dots, i_{n-1})$ of $C_z^{(i_1,\dots,i_{n-1})}$ some $n-p-1$ are numerating points on the corresponding coordinate axis, while other p are numbering intervals between points. For simplicity of notations, let us assume that in $C_z^{(i_1,\dots,i_{n-1})}$ indices $i_1, \dots, i_{n-p-1}$ are numbering points and $i_{n-p}, \dots, i_{n-1}$ are numbering intervals. In what follows we shall drop the upper multi-index in C_z.

Let $t = (t_1, \dots, t_{p+1}) \in (0,1)^{p+1}$ and suppose that $F(t) = (x_1, \dots, x_n)$. For any $r \in \{1, \dots, n\}$, let $j = n - r$. Each coordinate $x_r, 1 \leq r \leq n$, of $F(t)$ will be defined by induction on j.

For the base case $j = 0$, define $x_n = t_{p+1}(w_n'' - w_n') + w_n'$.

Suppose that points $x_n, x_{n-1}, \dots, x_{n-j+1}$ have allready been defined, for some j, $0 < j < p+1$. We shall now define the point x_{n-j}.

Consider the intersection

$$C_{x_n} \cap \{X_{n-1} = x_{n-1}, \dots, X_{n-j+1} = x_{n-j+1}\},$$

which, as we had assumed (the inductive hypothesis of the external induction), is a $(p-j+1)$-dimensional cell in $\{X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_{n-j+1} = x_{n-j+1}\}$, having multi-index $(i_1, \dots, i_{n-j})$. Here i_{n-j} numerates, as we supposed, an interval on X_{n-j}-coordinate axis. Denote the endpoints of this interval by w_{n-j}', w_{n-j}'' where $w_{n-j}' < w_{n-j}''$.

Now let $x_{n-j} = t_{p-j+1}(w_{n-j}'' - w_{n-j}') + w_{n-j}'$.

For $j = p$, we shall define the point $x_{n-p} = t_1(w_{n-p}'' - w_{n-p}') + w_{n-p}'$.

Because the dimension $\dim(C) = p+1$, the values $x_{n-p-1}, \dots, x_1$ of all the rest of the coordinates $X_{n-p-1}, \dots, X_1$ are defined uniquely.

Let us prove that map F is a homeomorphism. Clearly F is bijective.

Proceed by induction on p, the base case $p = 0$ being trivial, to show that the map F is continuous.

Let the points $t' = (t_1', \dots, t_{p+1}')$, $t'' = (t_1'', \dots, t_{p+1}'') \in (0,1)^{p+1}$ be infinitely close, i.e., the distance $\|t' - t''\|$ be infinitesimal (rel. to $\mathbb{R}_n$). Denote

$$x' = (x_1', \dots, x_n') = F(t'), \qquad x'' = (x_1'', \dots, x_n'') = F(t'').$$

We have to show that the distance $\|x' - x''\|$ is also infinitesimal.

Let us conduct one more induction on $j = n - r$, with the base $j = 0$, to prove that $|x_r' - x_r''|$, $1 \leq r \leq n - p$, are infinitesimals.

By the definition,

$$x'_n = t'_{p+1}(w''_n - w'_n) + w'_n, \quad x''_n = t''_{p+1}(w''_n - w'_n) + w'_n,$$

so $|x'_n - x''_n|$ is infinitesimal.

Suppose we have allready proved that $|x'_n - x''_n|, \ldots, |x'_{n-j+1} - x''_{n-j+1}|$ are all infinitesimals, for some j, $0 < j < p+1$. We shall now prove that $|x'_{n-j} - x''_{n-j}|$ is also infinitesimal.

Consider two intersections:

$$C_{x'_n} \cap \{X_{n-1} = x'_{n-1}, \ldots, X_{n-j+1} = x'_{n-j+1}\}$$

and

$$C_{x''_n} \cap \{X_{n-1} = x''_{n-1}, \ldots, X_{n-j+1} = x''_{n-j+1}\},$$

which are $(p-j+1)$-cells in $\{X_n = x'_n, X_{n-1} = x'_{n-1}, \ldots, X_{n-j+1} = x'_{n-j+1}\}$ and $\{X_n = x''_n, X_{n-1} = x''_{n-1}, \ldots, X_{n-j+1} = x''_{n-j+1}\}$ respectively, having the same multi-index $(i_1, \ldots, i_{n-j})$. Here, for each cell, index i_{n-j} numerates an open interval on X_{n-j}-axis. Let the endpoints of the interval for the first cell be w'_{n-j}, w''_{n-j} (with $w'_{n-j} < w''_{n-j}$) and for the second cell be $\hat{w}'_{n-j}$, $\hat{w}''_{n-j}$ (with $\hat{w}'_{n-j} < \hat{w}''_{n-j}$).

Consider the cells

$$C' = \bigcup_{z \in (w'_n, w''_n)} C_z^{(i'_1, \ldots, i'_{n-1})}, \qquad C'' = \bigcup_{z \in (w'_n, w''_n)} C_z^{(i''_1, \ldots, i''_{n-1})}$$

where

$$i'_1 = \cdots = i'_{n-j-1} = 0,\ i'_{n-j} = i_{n-j} - 1,\ i'_{n-j+1} = i_{n-j+1},\ \ldots,\ i'_{n-1} = i_{n-1}$$

and

$$i''_1 = \cdots = i''_{n-j-1} = 0,\ i''_{n-j} = i_{n-j} + 1,\ i''_{n-j+1} = i_{n-j+1},\ \ldots,\ i''_{n-1} = i_{n-1}.$$

By the inductive hypothesis of the induction on p, both C' and C'' are p-cells in $L^n_{n-j-1}[0] \subset I^n$ (for an example, see Figure 4-10), and so

$$||(w'_{n-j}, x'_{n-j+1}, \ldots, x'_n) - (\hat{w}'_{n-j}, x''_{n-j+1}, \ldots, x''_n)||$$

and

$$||(w''_{n-j}, x'_{n-j+1}, \ldots, x'_n) - (\hat{w}''_{n-j}, x''_{n-j+1}, \ldots, x''_n)||$$

are both infinitesimals. In particular, the values $|w'_{n-j} - \hat{w}'_{n-j}|$, $|w''_{n-j} - \hat{w}''_{n-j}|$ are infinitesimals. Since

$$x'_{n-j} = t'_{p-j+1}(w''_{n-j} - w'_{n-j}) + w'_{n-j}, \qquad x''_{n-j} = t''_{p-j+1}(\hat{w}''_{n-j} - \hat{w}'_{n-j}) + \hat{w}'_{n-j},$$

it follows that the difference

$$|x'_{n-j} - x''_{n-j}| \leq |t'_{p-j+1}(w''_{n-j} - w'_{n-j}) - t''_{p-j+1}(\hat{w}''_{n-j} - \hat{w}'_{n-j})| + |w'_{n-j} - \hat{w}'_{n-j}|$$

$$\leq |t'_{p-j+1} - t''_{p-j+1}|(w''_{n-j} - w'_{n-j}) + t''_{p-j+1}(|\hat{w}''_{n-j} - w''_{n-j}| + |\hat{w}'_{n-j} - w'_{n-j}|) + |w'_{n-j} - \hat{w}'_{n-j}|,$$

and thus, is infinitesimal. In particular, taking $j = p$ we prove that $|x'_{n-p} - x''_{n-p}|$ is infinitesimal.

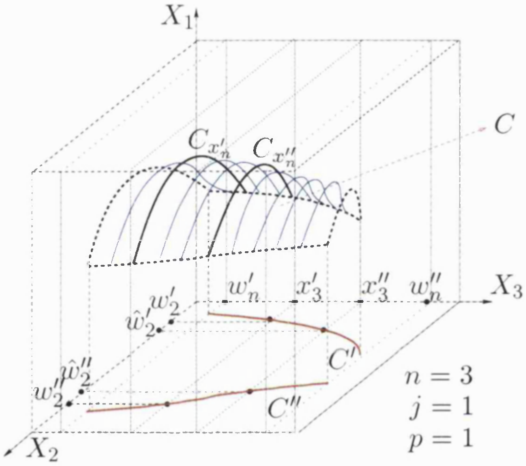


Figure 4-10: Establishing the continuity of the map F.

Finally, we have to show that for all $1 \leq r \leq n-p-1$ the absolute value $|x'_r - x''_r|$ is infinitesimal. Suppose the opposite. Then for a certain $1 \leq r_0 \leq n-p-1$ the value $|x'_{r_0} - x''_{r_0}|$ is not infinitesimal. Consider the limit point

$$\lim_{t'' \to t'} (x'') = \tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{n-p-1}, x'_{n-p}, \dots, x'_n),$$

where $\tilde{x}_{r_0} \neq x'_{r_0}$.

Note that for every r, $1 \leq r \leq n-p-1$, the value x'_r of X_r-coordinate has the number (in the alternating numbering of points and intervals on X_r-axis) corresponding to a point (not to an interval) and this number coincides with the index i_r of the multi-index for C. The same is true for the value $\tilde{x}_r$. We obtain the contradiction, since $\tilde{x}_{r_0} \neq x'_{r_0}$.

This finishes the proof of the continuity of the map F. The continuity of the inverse map

$$F^{-1} : C \longrightarrow (0,1)^{p+1}$$

can be established by the same proof in reverse.

Now we prove that the cell C is cylindrical.

Let C have a multi-index $(i_1, \dots, i_n)$ and we assume that i_1 is even (numbering a point and not an interval). Observe that the described cell decomposition of I^n induces a cell dccomposition of the set $L_1^n[0] \subset I^n$. Namely, C induces the cell C' in $\{X_1 = 0\}$ described by the multi-index $(0, i_2, \dots, i_n)$ by induction as follows. If i_n numerates a special value $x_n \in \rho_n(\Omega_s^{(n-1)})$ of the curve $\Gamma^{(n-1)}$ (rel. to X_n), then C' belongs to the hyperplane $\{X_n = x_n\}$; if i_n numerates an interval on $L_{n-1}^n[0]$, between two points, say x_n and y_n from $\rho_n(\Omega_s^{(n-1)})$, with $x_n < y_n$, then $C' \subset \{x_n < X_n < y_n\}$. Without loss of generality, we consider just the latter case. By the inductive hypothesis, for any $w \in (x_n, y_n)$, a cell with the multi-index $(0, i_2, \dots, i_{n-1})$ in $L_1^n[0] \cap \{X_n = w\}$ is defined. Let the dimension of the cell be p. As above, one can show that the union of $(0, i_2, \dots, i_{n-1})$-cells for all values $w \in (x_n, y_n)$ is a $(p+1)$-dimensional cell in $L_1^n[0]$. Obviously, C' is the bijective projection of C onto $\{X_1 = 0\}$. It follows that C is the graph of a continuous function with the domain C', taking values in axis X_1. Notice that the cells with indices $(i_1+1, i_2, \dots, i_n)$ and $(i_1-1, i_2, \dots, i_n)$ are $(p+2)$-dimensional sectors over C'.

It remains to prove that C' is a *cylindrical* cell. Consider a cell C'' in $L_2^n[0]$ described by the multi-index $(0, 0, i_3, i_4, \dots, i_n)$ in a way similar to the description of C'. Induction shows that C'' is indeed a cylindrical cell. Obviously, C'' is the bijective projection of C' onto $\{X_1 = X_2 = 0\}$. If i_2 is even numerating a special point on X_2, then C' is the graph of a continuous function with the domain C'', taking values in axis X_2. If i_2 numerates an interval on X_2, then C' is a set of points strictly between the graphs of two continuous functions with common domain C'', taking values in axis X_2. □

Corollary 4.3.4. *The union of cells described in Section §4.2 constitutes a cylindrical cell decomposition of I^n compatible with V, which we will denote by $\mathcal{D}$.*

Let in the description of $\mathcal{D}$, on the induction step i, $0 \le i \le n-1$, for fixed values of variables $X_{i+2}, \dots, X_n$, a new arbitrary point $x \in V$ be added to $\Omega_s^{(i)}$ (in addition to all special points of $\Gamma^{(i)}$, relative to X_{i+1}). Denote this new decomposition by $\mathcal{D}'$. As a consequence of the correctness of the cell decomposition description we also have:

Corollary 4.3.5. *The decomposition $\mathcal{D}'$ is a refinement of $\mathcal{D}$.*

Chapter 5

Complements of subanalytic sets

In this chapter we prove that for a subanalytic set determined by members of a given family of restricted real analytic functions, its complement within the unit cube is again a subanalytic set, determined by real analytic functions from the algebra generated by members of the same family, their partial derivatives, constants 0 and 1, and coordinate functions.

In Model theoretic terms this is equivalent to the model-completeness of the expansion of the real ordered field by a subalgebra $\mathcal{F}$ of restricted analytic functions closed under taking partial derivatives (Theorem 3.2.6). Using the terminology we have introduced in previous chapters, this result can be re-stated as follows: the complement (within the unit cube) of a $\mathcal{F}$-subanalytic set is $\mathcal{F}$-subanalytic. This is by no means a new result (see the discussion in Section §3.2); it was first proved by Gabrielov in [Gab96] and some generalizations already appeared in [Max98]. In fact, in our proof we actually use one of Gabrielov's preliminary results from [Gab96], which asserts that the closure and frontier within the unit cube of a $\mathcal{F}$-semianalytic set is $\mathcal{F}$-semianalytic (Lemma 3.2.4).

Our contribution is the introduction of a different and more elementary technique for proving Gabrielov's type of complement theorem in this setting. The complement theorem for a specific class of restricted analytic functions, immediately follows from the existence of a cylindrical cell decomposition of the unit cube, compatible with any given semianalytic set described by functions from this class, provided that all of its cells are defined by existential formulas involving analytic functions from the same class. We recursively construct the decomposition $\mathcal{D}$, described in Section §4.2, of the unit cube $I^n \subset \mathbb{R}^n_n$ that is compatible with a given $\mathcal{F}$-semianalytic set S defined in a neighbourhood G of I^n, in such a way that all of its cells are indeed $\mathcal{F}$-subanalytic. The idea is to construct the finite set $\Omega_0^{(n-1)}$ and thereby the cell decomposition $\mathcal{D}^{(n-1)}$

induced by $\mathcal{D}$ on $I^1 \subset \{X_1 = \cdots = X_{n-1} = 0\}$. For each cell C of $\mathcal{D}^{(n-1)}$ we determine the finite set $\Omega_0^{(n-2)}$ parameterized by the points of C, and thereby the cell decomposition $\mathcal{D}^{(n-2)}$ induced on $I^2 \subset \{X_1 = \cdots = X_{n-2} = 0\}$. On the last step of this recursion the parameterized set $\Omega_0^{(0)}$ is constructed, and the cell decomposition $\mathcal{D} = \mathcal{D}^{(0)}$ of I^n can be defined.

A straightforward representation of a finite (generally parametric) set of the kind $\Omega_0^{(i)}$, for some i, $0 \leq i \leq n-1$, by means of a $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(i+1)}$-formula $\Phi^{(i)}$ (see Section §4.2), would require quantifier alternation which we clearly want to avoid. In order to prove the main theorem, we obviously need to build *existential* $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(i+1)}$-formulas (that is, first-order formulas involving only real analytic functions from the collection $\mathcal{F}$ and constant symbols for each element of $\mathbb{R}_{i+1}$, in which no universal quantifiers appear) defining $\Omega_0^{(i)}$, $0 \leq i \leq n-1$. This is the purpose of the inductive construction presented in Section §5.1.

Once we have determined that the sets $\Omega^{(i)}$, $0 \leq i \leq n-1$, are $\mathcal{F}$-subanalytic it is not difficult to define each cell of the cylindrical decomposition $\mathcal{D}$ by some existential $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(i+1)}$-formula and thus ensuring that $\mathcal{D}$ comprises of $\mathcal{F}$-subanalytic sets. This is achieved by an inductive procedure, which we describe in Section §5.2. At step k, $0 \leq k \leq n-1$, of this procedure, we proceed in a straightforward manner to define each cell of the decomposition $\mathcal{D}^{(n-1-k)}$ of the cube $I^{k+1} = I^n \cap \{X_1 = \cdots = X_{n-1-k} = 0\}$ by utilizing existential formulas enumerating points in the finite set $\Omega^{(n-1-k)}$, parameterized by points belonging to cells in the decomposition $\mathcal{D}^{(n-k)}$ (which can be obtained by inductive hypothesis).

5.1 Representing $\Omega^{(i)}$ by an existential expression

Consider the sequence of ordered fields $\mathbb{R} = \mathbb{R}_0 \subset \mathbb{R}_1 \subset \cdots \subset \mathbb{R}_n \subset \mathbb{R}_{n+1}$, and positive elements with ordering $\varepsilon_0 \gg \varepsilon_1 \gg \cdots \gg \varepsilon_{n-1} \gg \delta$, such that $\varepsilon_i \in \mathbb{R}_{i+1}$ infinitesimal relative to $\mathbb{R}_i$, $0 \leq i \leq n-1$ and $\delta \in \mathbb{R}_{n+1}$ infinitesimal relative to $\mathbb{R}_n$.

Let $S \subset I^n \subset \mathbb{R}_n^n$ be a $\mathcal{F}$-semianalytic set and let $W = \rho_k(S) \subset I^{n-k+1}$, $1 \leq k \leq n$, where ρ_k denotes the projection map which omits the first $k-1$ coordinates. In order to prove that the set $\widetilde{W} = I^{n-k+1} \setminus W$ is $\mathcal{F}$-subanalytic it suffices to show the existence of a cylindrical cell decomposition D of the unit cube I^n compatible with S, so that each cell of D is $\mathcal{F}$-subanalytic. Indeed in this case, by the definition of a cylindrical cell decomposition, $D = D^{(0)}$ induces a cell decomposition $D^{(k-1)}$ of the cube $I^{n-k+1} = \rho_k(I^n) = \{X_1 = \cdots = X_{k-1} = 0\} \subset \mathbb{R}_n^{n-k+1}$ compatible with W. So $\widetilde{W}$ can be identified with a finite union of some cells of $D^{(k-1)}$ and thus is $\mathcal{F}$-subanalytic.

Referring back to the decomposition $\mathcal{D}$ described in Section §4.2 of the previous

Chapter, it turns out that the requirement that each cell of $\mathcal{D}$ is $\mathcal{F}$-subanalytic follows easily from the construction of existential $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(i+1)}$-formulas defining the sets of the kind $\Omega_0^{(i)}$, $0 \le i \le n-1$. In this section we describe an inductive procedure for building such existential formulas. This is the descending phase of our method, and it consists of n steps. At step i of this phase, for fixed values of the coordinate functions $X_{i+2}, \dots, X_n$, we construct the set $\Omega^{(i)}$ as follows.

- First, we approximate the set $\rho_{i+1}(\Omega_s^{(i)})$ of special values of the parametric curve $\Gamma^{(i)}$ (obtained by inductive hypothesis) relative to X_{i+1}-coordinate, with a finite parametric set of points defined by an *existential* $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(i+1)}$-formula;
- Then we define $\rho_{i+1}(\Omega_s^{(i)})$ by "passing to limit" (according to Lemma 3.4.3);
- Finally using the defining formula for $\rho_{i+1}(\Omega_s^{(i)})$ we build an existential $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(i+1)}$-formula $\Phi^{(i)}$ which determines $\Omega^{(i)}$.

Before we actually do that, we try to sketch some of the main ideas (without being too accurate) with the help of an example for $n = 2$. For the sake of simplicity, suppose that $f \in \mathcal{F}$ is an analytic function in the vicinity of I^2 and the set $V = \{f(X_1, X_2) = 0\} \subset I^2 \subset \mathbb{R}^2$ is such that $V \cap \{X_2 = \omega\}$ is finite for any $\omega \in [0,1]$.

Introduce a new variable Z and define the function

$$h(X_1, X_2, Z) = h_1(X_1, X_2, Z) \cdot h_2(X_1, X_2, Z),$$

where

$$h_1(X_1, X_2, Z) = f(X_1, X_2) - Z)^2 + \left(\frac{\partial f(X_1, X_2)}{\partial X_1}\right)^2$$

and

$$h_2(X_1, X_2, Z) = f(X_1, X_2)^2 + f(X_1 - Z, X_2)^2.$$

Let $\delta > 0$ be an element infinitesimal rel. to $\mathbb{R}$. Since δ is transcendental over $\mathbb{R}$, the set $V_1 = \{f(X_1, X_2) = \delta\}$ is a smooth hypersurface, on which the gradient vector of f does not vanish. If δ is viewed as infinitesimal then each point of V is infinitely close to a point of V_1. The function $h_1(X_1, X_2, \delta)$ defines a finite set of points on the hypersurface V_1 whose gradient is parallel to the X_2-axis. This is precisely the set of critical points of the projection of V_1 on $\{X_1 = 0\}$.

Points satisfying the equation $h_2(X_1, X_2, \delta) = 0$ correspond to ramification points of V. More precisely, if $x \in V$ is a ramification point of V, then the intersection of V with its infinitesimal shift V_2 along the axis X_1 is a finite set containing a point y such that $\|x - y\|$ is infinitesimal.

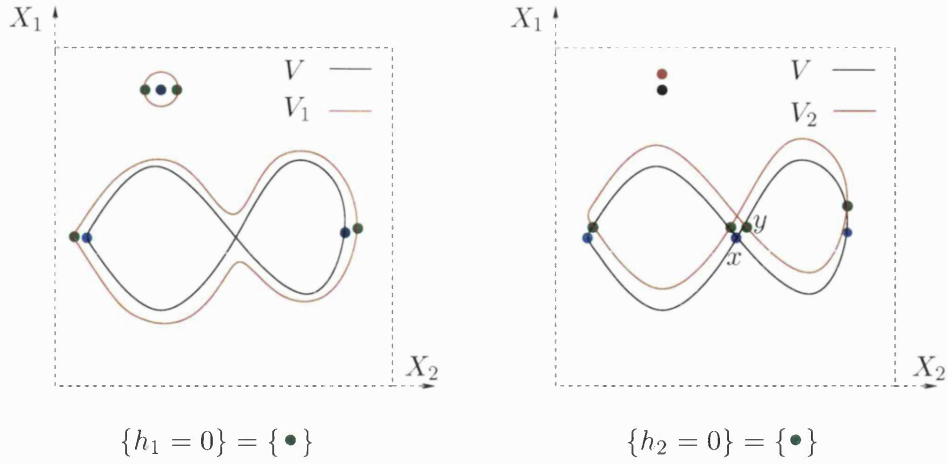


Figure 5-1: Approximating the special points (rel. to X_2) of a curve in $\mathbb{R}^2$.

Thus, the equation $h(X_1, X_2, \delta) = 0$ defines infinitesimal approximations to the points of the set $\Omega_s^{(1)}$ of all special points of V relative to X_2. The standard part of this set (see Proposition 3.4.3) can be defined by the quantifier-free formula $\Theta = \Theta_1 \vee \Theta_2$, where

$$\Theta_j := (Z = 0) \wedge \mathcal{C}\big((h_j(X_1, X_2, Z) = 0) \wedge (Z > 0)\big), \qquad j = 1, 2.$$

By Lemma 3.2.4, the set defined by Θ is indeed $\mathcal{F}$-semianalytic. Figure 5-2 gives an illustration of this for $j = 2$.

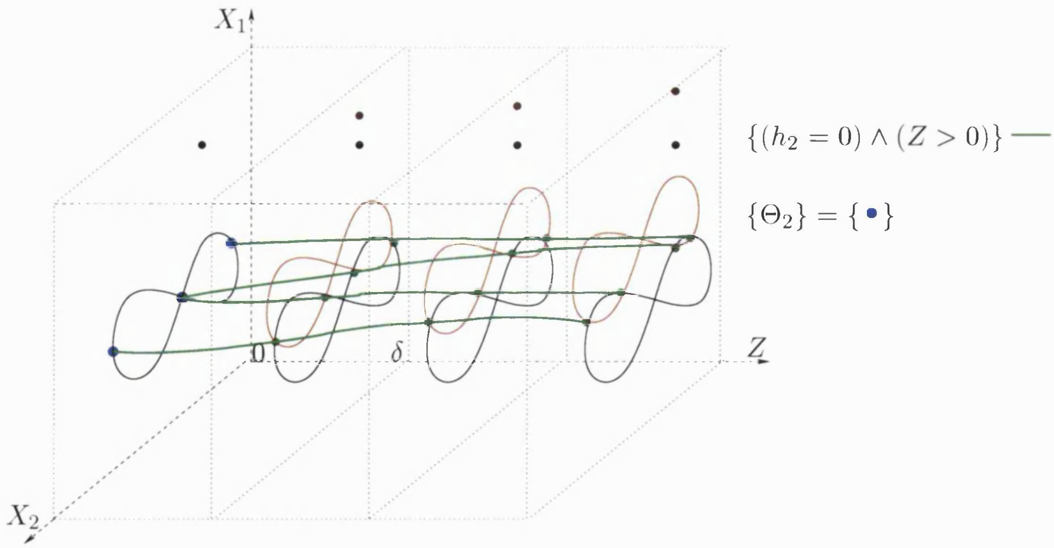


Figure 5-2: Obtaining the ramification points of a curve by "passing to limit".

Introduce a new variable Y_1. It follows, that the set $\Omega^{(1)}$ can be defined by the existential formula

$$\exists Y_1 \left(G_0(Y_1, X_1, X_2) \vee G_1(Y_1, X_1, X_2) \right),$$

where

$$G_0(Y_1, X_1, X_2) := \big((f(X_1, X_2) = 0 \vee (X_1 = 0) \vee (X_1 = 1)\big) \wedge \Theta(Y_1, X_2)$$

and

$$G_1(Y_1, X_1, X_2) := \big((f(X_1, X_2) = 0 \vee (X_1 = 0) \vee (X_1 = 1)\big) \wedge \Theta(Y_1, X_2 \pm \varepsilon_1).$$

To be more precise, the equation $f(X_1, X_2) = 0$ in the above formula defines a surface (which is a cylinder along Y_1 over $V \subset \{Y_1 = 0\}$) in the 3-dimensional space equipped with coordinates Y_1, X_1, X_2. On the other hand the formula $\Theta(Y_1, X_2)$ defines a finite collection of lines parallel to X_1-axis, over the set of special points rel. to X_2 of the curve $W = \{f(Y_1, X_2) = 0\} \cap \{X_1 = 0\}$. Let (y_1, x_2) be a special point rel. to X_2 of the curve W. The set $V_{y_1} = \{f(X_1, X_2) = 0\} \cap \{Y_1 = y_1\}$ is clearly, a copy of the curve V and so x_2 is actually a special value rel. to X_2 of V_{y_1}. So points in the intersection $V_{y_1} \cap \{X_2 = x_2\}$ will be defined by the formula $G_0(Y_1, X_1, X_2)$. Similarly points in the intersection $V_{y_1} \cap \{X_2 = x_2 \pm \varepsilon_1\}$ will be defined by $G_1(Y_1, X_1, X_2)$. Clearly, the projection of all these points onto $\{Y_1 = 0\}$, coincide with points in the set $\Omega^{(1)}$, see Figure 5-3.

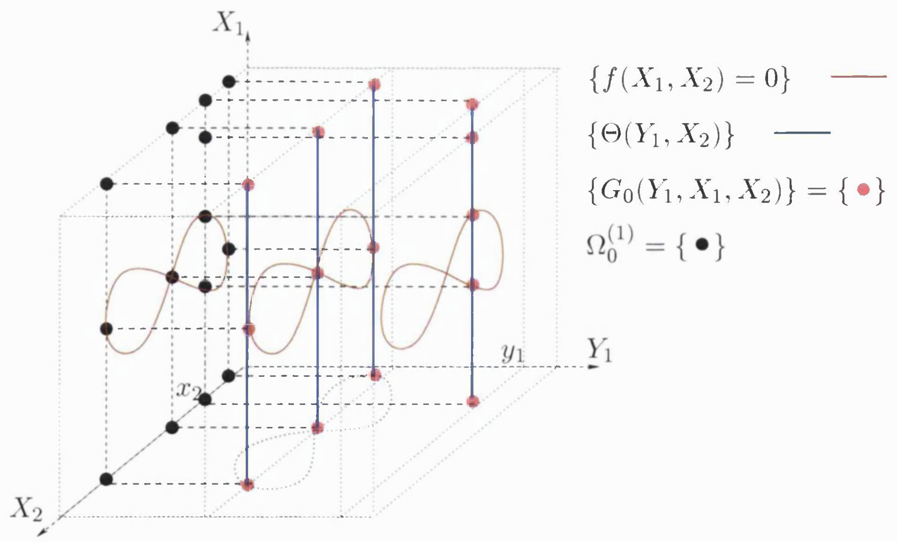


Figure 5-3: Defining points from $\Omega_0^{(1)}$.

Consider for a moment the case $n = 3$ and suppose that we have constructed an existential formula $\exists Y_1\, G(Y_1, X_1, X_2, X_3)$ defining the curve $\Gamma^{(2)}$. We can define, more or less in the same way as we did for $n = 2$, a finite set of points in the space of coordinates Y_1, X_1, X_2, X_3, whose projection onto $\{Y_1 = X_1 = X_2 = 0\}$ includes all X_3-coordinates of points of local maxima of the coordinate function X_3 on $\Gamma^{(2)}$. But if we insist in obtaining a *finite* set of points in some larger space whose projection onto X_3-axis includes all ramification values of $\Gamma^{(2)}$ (rel. to X_3) then we are forced to introduce one more new variable, say Y_2. The first step in this direction is to consider an "approximation" set of points in the space of all coordinates Y_1, Y_2, X_1, X_2, X_3 (that is, a set whose projection on X_3-axis includes points infinitely close to each ramification value of $\Gamma^{(2)}$), defined by the formulas

$$G(Y_1, X_1, X_2, X_3) \wedge G(Y_2, X_1 - \delta, X_2, X_3), \quad G(Y_1, X_1, X_2, X_3) \wedge G(Y_2, X_1, X_2 - \delta, X_3)$$

as the points of intersection of the set determined by $G(Y_1, X_1, X_2, X_3)$ with its infinitesimal δ-shifts along both X_1 and X_2. The standard part of this finite set of "approximation" points can then be computed as before. In order to define the set $\Omega^{(2)}$, again we have to introduce enough new variables so that the analogues of the formulas written out in the previous example for $n = 2$, also apply for this case.

5.1.1 A recursive construction of formulas

We now proceed to the actual description of the descending phase.

Define recursively the sequence of integers $s_0, \dots, s_{n-1}$ by setting $s_0 = 0$ and $s_{i+1} = 3s_i + i + 1$ for $0 \leq i \leq n-2$. Introduce new variables $Y_1, \dots, Y_{s_{n-1}}, Z$.

Define the expression $D_i(X) := (X_i \geq 0) \wedge (X_i \leq 1)$ so that $\{\bigwedge_{1 \leq i \leq k} D_i(X)\} = I^k$ and denote $T^{(m)} = (T_1, \dots, T_m)$, the m-tuple of variables T_i, $i \leq m$.

Let $X = X^{(n)}$. Introduce *quantifier-free* $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(i+1)}$-formulas $\widehat{G}^{(i)}(Y^{(s_i)}, X)$ by induction on i, $0 \leq i \leq n-1$, as follows. In steps $i = 0, 1$ we include comments explaining all non-trivial stages of the construction; formulas introduced in steps $i > 2$ can be interpreted analogously. Note that at step i we treat $X_{i+2}, \dots, X_n$ as parameters of these formulas.

Step i = 0.

$f^{(0)}(X) := \big(f(X)\big)^2$

In case when $f(X) = 0$ defines a finite set of points, then the equation $f^{(0)}(X) = 0$ defines the set $\Omega_0^{(0)} \setminus \{0, 1\}$ and possibly some points outside $(0, 1)$.

$h_Z^{(0)}(X, Z) := (f^{(0)}(X) - Z)^2$

The points satisfying $f^{(0)}(X) = 0$ are perturbed by Z.

$H_Z^{(0)}(X, Z) := \big(h_Z^{(0)} = 0\big) \vee \big(X_1(X_1 - 1) = 0\big)$

$\Theta_e^{(0)}(X, Z) := \mathcal{C}\big(H_Z^{(0)} \wedge (Z > 0)\big) \wedge (Z = 0)$

This formula defines the limits of perturbed points as $Z \to +0$, i.e., the set $\Omega_0^{(0)}$ and possibly some points outside $[0,1]$.

$\widehat{G}_0^{(0)}(X) := \Theta_e^{(0)} \wedge D_1(X)$
Defining the set $\Omega_0^{(0)}$ and the (parametric) curve $\widehat{\Gamma}_0^{(1)}$.

$\widehat{G}_1^{(0)}(X) := (f^{(0)}(X) = 0) \wedge \big(X_1 - \varepsilon_0)(X_1 - 1 + \varepsilon_0 = 0\big) \wedge (Z = 0)$
Defining the set $\Omega_1^{(0)}$ and the (parametric) curve $\widehat{\Gamma}_1^{(1)}$.

$\widehat{G}^{(0)}(X) := \widehat{G}_0^{(0)}(X) \vee \widehat{G}_1^{(0)}(X)$
Defining the set $\Omega^{(0)}$ and the (parametric) curve $\widehat{\Gamma}^{(1)}$.

Step i = 1.

$G_*^{(0)}(X) := \mathcal{C}\big(\widehat{G}_*^{(0)}(X)\big) \wedge D_2(X), \quad$ where $* \in \{0,1\}$.
This formula defines the curve $\Gamma_*^{(1)}$ (recall that $\mathcal{C}\Phi$ denotes the formula defining the topological closure of the set $\{\Phi\}$ defined by Φ).

$G^{(0)}(X) := G_0^{(0)}(X) \vee G_1^{(0)}(X) \equiv \bigvee_{1 \le l \le M_0} \big((f_l^{(0)}(X) = 0) \wedge (g_l^{(0)}(X) > 0)\big)$
Representing $G^{(0)}(X)$ as a Boolean combination of atomic equations and inequalities.

$h_{l,Z}^{(1)}(X, Z) := (f_l^{(0)}(X) - Z)^2 + \left(\frac{\partial f_l^{(0)}}{\partial X_1}\right)^2, \quad (1 \le l \le M_0)$
For small positive values of Z the equation $f_l^{(0)}(X) = Z$ defines a smooth hypersurface. Then $h_{l,Z}^{(1)}(X, Z) = 0$ defines the set of all critical points of the coordinate function X_2 on this hypersurface.

$H_Z^{(1)}(X, Z) := \bigvee_{1 \le l \le M_0} \big((h_{l,Z}^{(1)} = 0) \wedge (g_l^{(0)} > 0)\big)$
Collecting together the critical points on $f_l^{(0)}(X) = Z$ for all l, $1 \le l \le M_0$ and selecting those which are relevant. Note that for small values of $Z > 0$ all points of local extrema of the coordinate function X_2 on $\{G^{(1)}(X)\}$ are close to corresponding critical points.

$\Theta_e^{(1)}(X, Z) := \mathcal{C}\big(H_Z^{(1)} \wedge (Z > 0)\big) \wedge (Z = 0)$
Passing to limit as $Z \to +0$ produces a finite (parameterized) set of points on $\{G^{(0)}\}$ whose projection onto the X coordinates includes all points of local extrema of X_2 on the curve $\Gamma^{(1)}$.

$G_1^{(0)}(X_1 - Z, X) := G^{(0)}(X_1 - Z, X_2, \dots, X_n)$
This defines a curve obtained from $\{G^{(0)}\}$ by shifting it along the coordinate axis X_1 by Z.

$Q_Z^{(1)}(X_1 - Z, X) := G_1^{(0)} \wedge G^{(0)}$
Intersecting (the projection onto X coordinates of) $\{G^{(0)}\}$ with (the projection of) its shift produces a finite (parameterized) subset of $\Gamma^{(1)}$. Observe that for a small value $|Z|$ the value of the X_2-coordinate of each ramification point of $\Gamma^{(1)}$ is approximated by the corresponding values of the X_2-coordinate of some of the points from $\{Q_Z^{(1)}\}$.

$\Theta_r^{(1)}(X,Z) := \mathcal{C}\big(Q_Z^{(1)} \wedge (Z>0)\big) \wedge (Z=0)$
Passing to limit as $Z \to +0$ produces a finite (parameterized) set of points on $\{G^{(0)}\}$ such that its projection on the X coordinates contains all X_2-coordinates of ramification points of $\Gamma^{(1)}$.
$\Theta^{(1)}(X) := \Theta_e^{(1)} \vee \Theta_r^{(1)}$
Defining a finite set of points whose projection onto the X_2-axis contain all the special values of the curve $\Gamma^{(1)}$ (relative to X_2-coordinate).
$\widehat{G}_0^{(1)}(Y_1,X) := \widehat{G}^{(0)}(X) \wedge \Theta^{(1)}(Y_1,X_2,\dots,X_n)$
Defining a set whose projection along the variable Y_1 contains $\Omega_0^{(1)}$. Note that in the expression $\Theta^{(1)}(Y_1X_2,\dots,X_n)$ variable Y_1 stands for X_1. For any fixed values of parameters $X_3,\dots,X_n$ the set $\Theta^{(1)}$ is finite and therefore the set $\{\widehat{G}^{(0)} \wedge \Theta^{(1)}\}$ reduces to an intersection of two finite unions of affine subspaces of complementary dimensions in 3-dimensional space. It follows that $\widehat{G}_0^{(1)}(Y_1,X)$ is finite.
$\widehat{G}_1^{(1)}(Y_1,X) := \widehat{G}^{(0)}(X) \wedge \Theta^{(1)}(Y_1,X_2 \pm \varepsilon_1,X_3,\dots,X_n)$
Defining a finite set of points whose projection along variable Y_1 contains $\Omega_1^{(1)}$.
$\widehat{G}^{(1)}(Y_1,X) := \widehat{G}_0^{(1)} \vee \widehat{G}_1^{(1)}$
Defining a finite set of points whose projection along variables Y_1 contains $\Omega^{(1)}$.

Step i = 2.
$G_*^{(1)}(Y_1,X) := \mathcal{C}\big(\widehat{G}_*^{(1)}(Y_1,X)\big) \wedge D_3(X), \quad$ where $* \in \{0,1\}$.
$G^{(1)}(Y_1,X) := G_0^{(1)} \vee G_1^{(1)}$

$$\equiv \bigvee_{1\le l\le M_1} \big((f_l^{(1)}(Y_1,X)=0) \wedge (g_l^{(1)}(Y_1,X)>0)\big),$$

$h_{l,Z}^{(2)}(Y^{(2)},X,Z) := \big(f_l^{(1)}(Y_1,X)-Z\big)^2 + \Big(\frac{\partial f_l^{(1)}}{\partial X_1}\Big)^2 + \Big(\frac{\partial f_l^{(1)}}{\partial X_2}\Big)^2 + \Big(\frac{\partial f_l^{(1)}}{\partial Y_1}\Big)^2 + Y_2^2,\ (1 \le l \le M_1)$
$H_Z^{(2)}(Y^{(2)},X,Z) := \bigvee_{1\le l\le M_1} \big((h_{l,Z}^{(2)}=0) \wedge (g_l^{(1)}>0)\big)$
$\Theta_e^{(2)}(Y^{(2)},X,Z) := \mathcal{C}\big(H_Z^{(2)} \wedge (Z>0)\big) \wedge (Z=0)$
$G_1^{(1)}(Y_2,X_2-Z,X) := G^{(1)}(Y_2,X_1,X_2-Z,X_3,\dots,X_n)$
$G_2^{(1)}(Y_2,X_1-Z,X) := G^{(1)}(Y_2,X_1-Z,X_2,\dots,X_n)$
$Q_{1,Z}^{(2)}(Y^{(2)},X_2-Z,X) := G_1^{(1)} \wedge G^{(1)}$
$Q_{2,Z}^{(2)}(Y^{(2)},X_1-Z,X) := G_2^{(1)} \wedge G^{(1)}$
$Q_Z^{(2)}(Y^{(2)},X_1-Z,X_2-Z,X) := Q_{1,Z}^{(2)} \vee Q_{2,Z}^{(2)}$
$\Theta_r^{(2)}(Y^{(2)},X,Z) := \mathcal{C}\big(Q_Z^{(2)} \wedge (Z>0)\big) \wedge (Z=0)$

$\Theta^{(2)}(Y^{(2)}, X) := \Theta_e^{(2)} \vee \Theta_r^{(2)}$

$\widehat{G}_0^{(2)}(Y^{(5)}, X) := \widehat{G}^{(1)}(Y_3, X) \wedge \Theta^{(2)}(Y_1, Y_2, Y_4, Y_5, X_3, \ldots, X_n)$

$\widehat{G}_1^{(2)}(Y^{(5)}, X) := \widehat{G}^{(1)}(Y_3, X) \wedge \Theta^{(2)}(Y_1, Y_2, Y_4, Y_5, X_3 \pm \varepsilon_2, X_4, \ldots, X_n)$

$\widehat{G}^{(2)}(Y^{(5)}, X) := \widehat{G}_0^{(2)} \vee \widehat{G}_1^{(2)}$

General step: On the **step i**, for $i \leq n-2$, define

$$\widehat{G}_0^{(i)}(Y^{(s_i)}, X), \qquad \widehat{G}_1^{(i)}(Y^{(s_i)}, X).$$

On the **step (i + 1)**

$G_*^{(i)}(Y^{(s_i)}, X) := \mathcal{C}\big(\widehat{G}_*^{(i)}(Y^{(s_i)}, X)\big) \wedge D_{i+2}(X), \quad$ where $* \in \{0, 1\}$.

$G^{(i)}(Y^{(s_i)}, X) := G_0^{(i)} \vee G_1^{(i)}$

$\qquad \equiv \bigvee_{1 \leq l \leq M_i} \big((f_l^{(i)}(Y^{(s_i)}, X) = 0) \wedge \big(g_l^{(i)}(Y^{(s_i)}, X) > 0\big)\big)$

$h_{l,Z}^{(i+1)}(Y^{(2s_i)}, X, Z) := \big(f_l^{(i)} - Z\big)^2 + \sum_{1 \leq j \leq i+1} \Big(\frac{\partial f_l^{(i)}}{\partial X_j}\Big)^2 +$

$\qquad + \sum_{1 \leq j \leq s_i} \Big(\frac{\partial f_l^{(i)}}{\partial Y_j}\Big)^2 + \sum_{1+s_i \leq j \leq 2s_i} \Big(Y_j\Big)^2, \quad (1 \leq l \leq M_i)$

$H_Z^{(i+1)}(Y^{(2s_i)}, X, Z) := \bigvee_{1 \leq l \leq M_i} \big((h_{l,Z}^{(i+1)} = 0) \wedge (g_l^{(i)} > 0)\big)$

$\Theta_e^{(i+1)}(Y^{(2s_i)}, X, Z) := \mathcal{C}\big(H_Z^{(i+1)} \wedge (Z > 0)\big) \wedge (Z = 0)$

$G_1^{(i)} := G^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1, \ldots, X_i, X_{i+1} - Z, X_{i+2}, \ldots, X_n)$

$G_2^{(i)} := G^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1, \ldots, X_{i-1}, X_i - Z, X_{i+1}, \ldots, X_n)$

....

$G_j^{(i)} := G^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1, \ldots, X_{i+1-j}, X_{i+2-j} - Z, X_{i+3-j}, \ldots, X_n), \ 1 \leq j \leq i+1$

....

$G_{i+1}^{(i)} := G^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1 - Z, X_2, \ldots, X_n)$

$Q_{1,Z}^{(i+1)}(Y^{(2s_i)}, X_{i+1} - Z, X) := G_1^{(i)} \wedge G^{(i)}$

$Q_{2,Z}^{(i+1)}(Y^{(2s_i)}, X_i - Z, X) := G_2^{(i)} \wedge G^{(i)}$

....

$Q_{j,Z}^{(i+1)}(Y^{(2s_i)}, X_{i+2-j} - Z, X) := G_j^{(i)} \wedge G^{(i)}, \quad 1 \leq j \leq i+1$

....

$Q_{i+1,Z}^{(i+1)}(Y^{(2s_i)}, X_1 - Z, X) := G_{i+1}^{(i)} \wedge G^{(i)}$

$Q_Z^{(i+1)}(Y^{(2s_i)}, X_1 - Z, \ldots, X_{i+1} - Z, X) := \bigvee_{1 \leq j \leq i+1} \big(Q_{j,Z}^{(i+1)}\big)$

$\Theta_r^{(i+1)}(Y^{(2s_i)}, X, Z) := \mathcal{C}\big(Q_Z^{(i+1)} \wedge (Z > 0)\big) \wedge (Z = 0)$

$\Theta^{(i+1)}(Y^{(2s_i)}, X) := \Theta_e^{(i+1)} \vee \Theta_r^{(i+1)}$

$$\begin{aligned}
\widehat{G}_0^{(i+1)}(Y^{(s_{i+1})}, X) &:= \widehat{G}^{(i)}(Y_{1+2s_i}, \dots, Y_{3s_i}, X) \wedge \\
&\qquad \Theta^{(i+1)}(Y^{(2s_i)}, Y_{3s_i+1}, \dots, Y_{3s_i+i+1}, X_{i+2}, \dots, X_n) \\
\widehat{G}_1^{(i+1)}(Y^{(s_{i+1})}, X) &:= \widehat{G}^{(i)}(Y_{1+2s_i}, \dots, Y_{3s_i}, X) \\
&\qquad \Theta^{(i+1)}(Y^{(2s_i)}, Y_{3s_i+1}, \dots, Y_{3s_i+i+1}, X_{i+2} \pm \varepsilon_{i+1}, \dots, X_n) \\
\widehat{G}^{(i+1)}(Y^{(s_{i+1})}, X) &:= \widehat{G}_0^{(i+1)} \vee \widehat{G}_1^{(i+1)}.
\end{aligned}$$

This finishes the description of the general step.

5.1.2 Proving the subanalyticity of $\Omega^{(i)}$

We want to show that the sets of the kind $\Omega^{(i)}$, $0 \le i \le n-1$, can actually be defined by existential formulas involving only analytic functions from the collection $\mathcal{F}$, and hence, that they are $\mathcal{F}$-subanalytic.

For each i, $0 \le i \le n-1$, let $\pi_i : \quad \mathbb{R}_{i+1}^{s_i+i+1} \longrightarrow \mathbb{R}_{i+1}^{i+1}$, be the projection map along $Y^{(s_i)}$, onto the space with coordinates $X_1, \dots, X_{i+1}$. Define

$$H_\delta^{(i+1)} := H_Z^{(i+1)} \wedge (Z = \delta) \quad \text{and} \quad Q_\delta^{(i+1)} := Q_Z^{(i+1)} \wedge (Z = \delta).$$

It follows directly from the definition of these sets and Proposition 3.4.3, that for any fixed values of variables $X_{i+2}, \dots, X_n$, the equalities

$$st_{i+1}(\{H_\delta^{(i+1)}\}) = \{\Theta_e^{(i+1)}\} \quad \text{and} \quad st_{i+1}(\{Q_\delta^{(i+1)}\}) = \{\Theta_r^{(i+1)}\}$$

hold in the space $\mathbb{R}_{i+1}^{2s_i+i+2}$ equipped with variables $Y^{(2s_i)}, X_1, \dots, X_{i+2}$ (recall that st_{i+1} denotes the standard part operation relative to the field $\mathbb{R}_{i+1}$).

Lemma 5.1.1. *For any index i, $0 \le i \le n-1$,*

1. *The set defined by the formula $\widehat{G}^{(i)}(Y^{(s_i)}, X)$ is $\mathcal{F}$-semianalytic.*
2. *The subset of $\mathbb{R}_{i+1}^{s_i+i+1}$ defined by the formula $\widehat{G}^{(i)}(Y^{(s_i)}, X_1, \dots, X_{i+1}, \omega)$ for any $\omega \in I^{n-i-1}$, consists of a finite number of points.*

Proof. The first part of the lemma follows directly from the actual definition of the formula $\widehat{G}^{(i)}(Y^{(s_i)}, X)$, since the family $\mathcal{F}$ of restricted analytic functions is closed under addition, multiplication and taking partial derivatives; and the closure of a $\mathcal{F}$-semianalytic is $\mathcal{F}$-semianalytic (Lemma 3.2.4).

We shall proceed by induction on i to prove the second part of the lemma.

First consider the case $i = 0$. Fix variables $X_2, \dots, X_n$. Notice that if the function $f(X)$ is identically zero, then $\{f^{(0)}(X) = \delta\} = \emptyset$. In this case $\{\widehat{G}_0^{(0)}\} = \{0, 1\}$ and

$\{\widehat{G}_1^{(0)}\} = \{\varepsilon_0, 1-\varepsilon_0\}$. If, on the other hand, $f(X)$ defines a finite collection of points then $H_\delta^{(0)}$ also defines a finite set of points, in particular, every point from $\{f^{(0)}(X)=0\}$ is infinitesimally close to a point in this set. By Lemma A.3 and Lemma A.8, we get that $\{\Theta_e^{(0)}\} = st_0\{H_\delta^{(0)}\} = \{f^{(0)}(X)=0\}$ and so $\{\widehat{G}^{(0)}\}$ is also finite.

We suppose that the lemma is true for $i < n-1$ and show that is also true for the case $i = n-1$.

We first prove that $\{\Theta_e^{(n-1)}\}$ consists of a finite number of points. By inductive hypothesis, we get that for every fixed value ω of the last coordinate X_n, the set $\{\widehat{G}^{(n-2)}(Y^{(s_{n-2})}, X_1, \dots X_{n-1}, \omega)\}$ is 0-dimensional in the space $\mathbb{R}_{n-1}^{s_{n-2}+n-1}$. Thus, the formula

$$G^{(n-2)} = \mathcal{C}(\widehat{G}^{(n-2)}(Y^{(s_{n-2})}, X)) := \bigvee_{1 \le l \le M_{n-2}} (f_l^{(n-2)} = 0) \wedge (g_l^{(n-2)} > 0),$$

defines a closed semianalytic curve in the space $\mathbb{R}_{n-1}^{s_{n-2}+n}$, equipped with variables $Y^{(s_{n-2})}, X$. Let $G_l^{(n-2)} = (f_l^{(n-2)} = 0) \wedge (g_l^{(n-2)} > 0)$.

Recall that for each l, $1 \le l \le M_{n-2}$,

$$h_{l,Z}^{(n-1)}(Y^{(2s_{n-2})}, X, Z) = \left(f_l^{(n-2)} - Z\right)^2 + \sum_{1 \le j \le n-1} \left(\frac{\partial f_l^{(n-2)}}{\partial X_j}\right)^2 + \\ + \sum_{1 \le j \le s_{n-2}} \left(\frac{\partial f_l^{(n-2)}}{\partial Y_j}\right)^2 + \sum_{1+s_{n-2} \le j \le 2s_{n-2}} \left(Y_j\right)^2.$$

Denote by $h_{l,\delta}^{(n-1)}$ the analytic function obtained by substituting $Z = \delta$ in the function $h_{l,Z}^{(n-1)}$. The function $f_l^{(n-2)}$ is definable over $\mathbb{R}_{n-1}$. The element δ, being infinitesimal relative to $\mathbb{R}_{n-1}$, cannot be a critical value of $f_l^{(n-2)}$ (see Corollary A.7). Hence $\{f_l^{(n-2)} - \delta = 0\}$ is a smooth hypersurface in the space $\mathbb{R}_{n-1}^{s_{n-2}+n}$ of coordinates $Y^{(s_{n-2})}, X$ on which the gradient vector of $f_l^{(n-2)}$ does not vanish. Moreover, for each point $x \in \{f_l^{(n-2)} = 0\}$, there exists a point $y \in \{f_l^{(n-2)} - \delta = 0\}$, such that the distance $||x-y||$ is infinitesimal relative to $\mathbb{R}_{n-1}$.

The function $h_{l,\delta}^{(n-1)}$ defines the (not necessarily zero dimensional) set of points on the hypersurface $\{f_l^{(n-2)} - \delta = 0\}$, whose gradient is parallel to the X_n-axis (equivalently, a point $p \in \{h_{l,\delta}^{(n-1)} = 0\}$ if and only if the hyperplane $\{X_n = p_n\}$ is the tangent space of $\{f_l^{(n-2)} - \delta = 0\}$ at p). The set of points of local extremum of X_n-coordinate on the hypersurface $\{f_l^{(n-2)} - \delta = 0\}$ is clearly a subset of $\{h_{l,\delta}^{(n-1)} = 0\}$.

Let $\rho_n' = \rho_n \circ \pi_{n-1}$ denote the projection on the last coordinate X_n. Then the set of points defined by the equation $h_{l,\delta}^{(n-1)} = 0$ is precisely the set of critical points of ρ_n' on $\{f_l^{(n-2)} - \delta = 0\}$.

Suppose that C is a definably connected component of the set $\{h_{l,\delta}^{(n-1)} = 0\}$. We want to show that $C \subset \{X_n = w\}$, for some $w \in \mathbb{R}_{n+1}$. If C consists of a single point then we are done, so we can assume that $dim(C) \geq 1$. Let the points p and q belong in C. Then there exists a definably connected curve $\Gamma_{p,q} \subset C \subset \{h_{l,\delta}^{(n-1)} = 0\}$, containing these points. Assume that $p_n \neq q_n$ (w.l.o.g., $p_n < q_n$). Then obviously $\Gamma_{p,q} \not\subset \{X_n = p_n\}$ and for all values w between p_n and q_n, the intersection $\Gamma_{p,q} \cap \{X_n = w\}$ is nonempty. Hence $\{p_n \leq X_n \leq q_n\} \subset \rho_n'(\{h_{l,\delta}^{(n-1)} = 0\})$. But Sard's Theorem (with the help of the transfer principle) implies that the set $\rho_n'(\{h_{l,\delta}^{(n-1)} = 0\})$ is a finite collection of points. This contradiction shows that $p_n = q_n$.

It follows, that there exists some $\lambda \in \mathbb{N}$ and elements $w_1, \dots, w_\lambda \in \mathbb{R}_{n+1}$, such that

$$\{h_{l,\delta}^{(n-1)} = 0\} \subset \{f_l^{(n-2)} - \delta = 0\} \cap \big\{(X_n = w_1) \vee \cdots \vee (X_n = w_\lambda)\big\}.$$

Introduce the $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(n-1)}$-formula $H_{l,\delta}^{(n-1)} := \big((h_{l,\delta}^{(n-1)} = 0) \wedge (g_l^{(n-2)} > 0)\big)$. Its defining set

$$\{H_{l,\delta}^{(n-1)}\} \subset \big\{(f_l^{(n-2)} - \delta = 0) \wedge (g_l^{(n-2)} > 0)\big\} \cap \big\{(X_n = w_1) \vee \cdots \vee (X_n = w_\lambda)\big\}.$$

Moreover, as a result of the continuity of analytic functions definable over the field $\mathbb{R}_{n-1}$, we get the inclusion $st_{n-1}\big(\{f_l^{(n-2)} - \delta = 0\}\big) \subset \{f_l^{(n-2)} = 0\}$, and thus, the set

$$st_{n-1}\big(\{H_{l,\delta}^{(n-1)}\}\big) \subset \{G_l^{(n-2)}\} \cap \big\{(X_n = st_{n-1}(w_1)) \vee \cdots \vee (X_n = st_{n-1}(w_\lambda))\big\}$$

is zero-dimensional. Let $\{H_\delta^{(n-1)}\}$ be the union of sets $\{H_{l,\delta}^{(n-1)}\}$, $1 \leq l \leq M_{n-2}$. Then $\{\Theta_e^{(n-1)}\} = st_{n-1}\big(\{H_\delta^{(n-1)}\}\big)$ consists of a finite number of points.

Secondly, we prove that $\{\Theta_r^{(n-1)}\} \subset \mathbb{R}_{n-1}^{2s_{n-2}+n}$ consists of a finite number of points.

Let

$$G_{j,\delta}^{(n-2)} := G_j^{(n-2)} \wedge (Z = \delta).$$

Due to the inductive hypothesis, the sets

$$G_j := \{G_{j,\delta}^{(n-2)}\} = \{G^{(n-2)}(Y_{1+s_{n-2}}, \dots, Y_{2s_{n-2}}, X_1, \dots, X_{n-1-j}, X_{n-j} - \delta, X_{n+1-j}, \dots, X_n)\}$$

and $G := \{G^{(n-2)}(Y^{(s_{n-2})}, X)\}$ are $\mathcal{F}$-semianalytic curves in the $(s_{n-2}+n)$-dimensional spaces of coordinates $Y_{1+s_{n-2}}, \dots, Y_{2s_{n-2}}, X$ and $Y^{(s_{n-2})}, X$ respectively. Consider the projections G', G_j' of G, G_j respectively on the subspace of coordinates $X_1, \dots, X_n$, which are also subanalytic curves. Observe that G_j' is obtained from G' by an infinitesimal (relative to $\mathbb{R}_{n-1}$) δ-shift along the coordinate X_{n-j}. We prove that the intersection $G' \cap G_j'$ consists of at most finite number of points. Indeed, suppose that $\dim(G' \cap G_j') = 1$. Then there is a point $x \in G' \cap G_j'$ defined over the field $\mathbb{R}_{n-1}$ not including δ. This contradicts to x being a δ-shift of another point defined over the field not including δ, namely of $G' \cap \{X_1 = x_1, \dots, X_{n-j-1} = x_{n-j-1}, X_{n-j+1} = x_{n-j+1}, \dots, X_n = x_n\}$. It follows that the set $\{Q^{(n-1)}_{j,\delta}\} = G \cap G_j$, if non-empty, is a transversal intersection of two s_{n-2}-dimensional planes in the $2s_{n-2}$-dimensional subspace of coordinates $Y^{(2s_{n-2})}$ and therefore is zero-dimensional. Moreover, $\pi_{n-1}(\{Q^{(n-1)}_{j,\delta}\}) = G' \cap G'_j$.

As a result, the set $\{Q^{(n-1)}_{\delta}\}$, which is the union of sets $\{Q^{(n-1)}_{j,\delta}\}$, $1 \leq j \leq n-1$, is zero-dimensional. By Proposition 3.4.3, $dim\big(st_{n-1}(\{Q^{(n-1)}_{\delta}\})\big) \leq dim\big(\{Q^{(n-1)}_{\delta}\}\big)$; it follows that the set $st_{n-1}\big(\{Q^{(n-1)}_{\delta}\}\big) = \{\Theta^{(n-1)}_{r}\}$ is finite. Clearly, the set defined by the formula

$$\Theta^{(n-1)}(Y^{(2s_{n-2})}, X) := \Theta^{(n-1)}_{e} \vee \Theta^{(n-1)}_{r},$$

also consists of a finite number of points.

We now prove that the set $\{\widehat{G}^{(n-1)}_{0}\} \subset \mathbb{R}^{3s_{n-2}+2n-1}_{n-1}$ consists of a finite number of points. According to the formula,

$$\widehat{G}^{(n-1)}_{0}(Y^{(s_{n-1}}, X_n) := \widehat{G}^{(n-2)}\big(Y_{1+2s_{n-2}}, \dots, Y_{3s_{n-2}}, X\big) \wedge$$
$$\Theta^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n).$$

Let $G_1 = \{\widehat{G}^{(n-2)} = 0\}$ and $G_2 = \{\Theta^{(n-1)}\}$. Notice that G_1 is a finite set of points in the $(s_{n-2}+n-1)$-dimensional space of coordinates $Y_{1+2s_{n-2}}, \dots, Y_{3s_{n-2}}, X_1, \dots, X_{n-1}$, while G_2 is a finite set of points in the $(2s_{n-2}+n)$-dimensional space of coordinates $Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n$. Now consider the projections G'_1 G'_2 of G_1, G_2 respectively on the subspace of coordinates $X_1, \dots, X_n$. G'_1 is a subanalytic curve along X_n, while G'_2 is a union of $(n-1)$-dimensional planes orthogonal to the X_n-coordinate axis. Thus, the set $G'_1 \cap G'_2$ is a finite collection of points. It follows that $G_1 \cap G_2$, if non empty, is a transversal intersection of two planes of complementary dimensions $(2s_{n-2}+n-1)$ and s_{n-2}, respectively, in the (s_{n-1})-dimensional subspace $\{X_1 = \cdots = X_n = 0\}$ of coordinates $Y_1, \dots, Y_{3s_{n-2}}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}$, and is therefore finite. Moreover

$$\pi_{n-1}(\{\widehat{G}^{(n-1)}_{0}\}) = G'_1 \cap G'_2.$$

Similarly one could prove that $\{\widehat{G}^{(n-1)}_{1} = 0\} \subset \mathbb{R}^{s_{n-1}+n}_{n}$ is zero dimensional. Then obviously, their union $\{\widehat{G}^{(n-1)}\}$ is also a finite collection of points. $\square$

It follows that for $i = 0, \dots, n-2$ and fixed values of variables $X_{i+3}, \dots, X_n$ the formula

$$G^{(i)} = \mathcal{C}\big(\widehat{G}^{(i)}(Y^{(s_i)}, X)\big) \wedge D_{i+2}(X) \equiv \bigvee_{1 \leq l \leq M_i} \big((f^{(i)}_l(Y^{(s_i)}, X) = 0) \wedge (g^{(i)}_l(Y^{(s_i)}, X) > 0)\big),$$

with the functions $f^{(i)}_l$ and $g^{(i)}_{l,1}, \dots, g^{(i)}_{l,J_i}$ belonging to the collection $\mathcal{F}$, defines a closed curve in the space $\mathbb{R}^{i+2+s_i}_{i+2}$.

In the next Lemma we make heavy use of the notation intoduced in Definition 4.2.1.

Lemma 5.1.2. *For index i, $0 \le i \le n-2$, and any $\omega \in I^{n-i-2}$,*

1. $\mathcal{E}_{i+2}\left(\pi_{i+1}(\{G^{(i)}(Y^{(s_i)}, X^{(i+2)}, \omega)\})\right) \subset \pi_{i+1}\left(\{\Theta_e^{(i+1)}(Y^{(2s_i)}, X^{(i+2)}, \omega)\}\right)$,
2. *For each $k \in \{1, \dots, i+2\}$,*

$$\rho_k\big(\pi_{i+1}(\{G^{(i)}(Y^{(s_i)}, X^{(i+2)}, \omega)\})\big) \subset \pi_{i+1}\big(\{G^{(i)}(Y^{(s_i)}, X^{(i+2)}, \omega)\}\big),$$

where, ρ_k denotes the projection onto the subspace $\{X_1 = \cdots = X_{k-1} = 0\}$ equipped with coordinates $X_k, X_{k+1} \dots, X_{i+2}$.

3. $\rho_{i+2}\big(\mathcal{R}_{i+2}\big(\pi_{i+1}(\{G^{(i)}(Y^{(s_i)}, X^{(i+2)}, \omega)\})\big)\big) \subset \rho_{i+2}\big(\pi_{i+1}\big(\{\Theta_r^{(i+1)}(Y^{(2s_i)}, X^{(i+2)}, \omega)\}\big)\big)$.

Proof. Without loss of generality, we prove the lemma for the case $i = n-2$.

(i) Observe that $\mathcal{E}_n\left(\pi_{n-1}(\{G^{(n-2)}\})\right) \subset \pi_{n-1}\left(\mathcal{E}_n(\{G^{(n-2)}\})\right)$, so in order to prove the first part of the lemma it suffices to show that $\mathcal{E}_n(\{G^{(n-2)}\}) \subset \{\Theta_e^{(n-1)}\}$.

Suppose that $y \in \mathcal{E}_n(\{G^{(n-2)}\})$, i.e., y is a point of local maximum or minimum of X_n-coordinate on $\{G^{(n-2)}\}$. Clearly, there must be some index j, such that $y \in \{f_j^{(n-2)} = 0\} \cap \{g_j^{(n-2)} > 0\}$. The function $f_j^{(n-2)}$ can not be identically zero, since in this case $\{G_j^{(n-2)}\}$ would have the full dimension. Suppose that C_y is the definably connected component of $\{f_j^{(n-2)} = 0\}$ that contains the point y. The open set $\{g_j^{(n-2)} > 0\}$ includes y and so there exists some sufficiently small $t \in \mathbb{R}_{n-1}$, $t > 0$, such that $B_y(t) \subset \{g_j^{(n-2)} > 0\}$, where $B_y(t)$ denotes the open ball of radius t, centered on y. Without loss of generality, suppose that y is a point of local maximum of X_n-coordinate on $\{G^{(n-2)}\}$. Recall that for fixed value of X_n, the set $\{G^{(n-2)}\}$ is finite. Let y_n be the value of the X_n-coordinate of the point y; then the set $B_y(t) \cap \{X_n \ge y_n\} \cap C_y$ consists only of the point y. This means that y is a local maximum of X_n-coordinate on $\{f_j^{(n-2)} = 0\}$ satisfying $g_j^{(n-2)}(y) > 0$ (similarly for local minimum).

Thus, in order to find the points of local extremum of the coordinate function X_n on $\{G^{(n-2)}\}$, it suffices to determine, for each j, $1 \le j \le M_{n-2}$, the points of local extremum of the coordinate function X_n on the zeroset of $f_j^{(n-2)}$ for which the function $g_j^{(n-2)}$ takes positive values.

For each such j, denote by $h_{j,\delta}^{(n-1)}$ the analytic function obtained by substituting $Z = \delta$ in the function $h_{j,Z}^{(n-1)}$, where δ is a positive element of $\mathbb{R}_{n+1}$, infinitesimal relative to $\mathbb{R}_n$. Recall that in the course of the proof of the fact that $\{\Theta_e^{(n-1)}\}$ consists of a finite number of points (see Lemma 5.1.1), we established that $\{f_j^{(n-2)} - \delta = 0\}$ is a hypersurface in the space $\mathbb{R}_{n+1}^{s_{n-2}+n}$ of coordinates $Y^{(s_{n-2})}, X^{(n)}$ and that the set of local extremum $\mathcal{E}_n(\{f_j^{(n-2)} - \delta = 0\}) \subset \{h_{j,\delta}^{(n-1)} = 0\}$.

Let $G_{j,\delta}^{(n-2)} := (f_j^{(n-2)} - \delta = 0) \wedge (g_j^{(n-2)} > 0)$ and $G_\delta^{(n-2)} := \bigvee_j G_{j,\delta}^{(n-2)}$. Since δ is infinitesimal relative to $\mathbb{R}_{n-1}$, there must be a definably connected component $C_y^{(\delta)}$

of $\{G_\delta^{(n-2)}\}$ containing a point z, such that $st_{n-1}(z) = y$, and for sufficiently small $0 < t \in \mathbb{R}_{n-1}$, either $C_y^{(\delta)} \cap \{X_n = y_n + \frac{t}{2}\} \cap B_y(t) = \emptyset$ (if y is a local maximum of X_n-coordinate on $\{G^{(n-2)}\}$), or $C_y^{(\delta)} \cap \{X_n = y_n - \frac{t}{2}\} \cap B_y(t) = \emptyset$ (if y is a local minimum of X_n-coordinate on $\{G^{(n-2)}\}$). Thus, the coordinate function X_n has a point of local extremum on $C_y^{(\delta)}$, say w, such that $st_{n-1}(w_n) = y_n$. It follows that $w \in \bigcup_{1 \le j \le M_{n-2}} \{(h_{j,\delta}^{(n-1)} = 0) \wedge (g_j^{(n-2)} > 0)\}$, and as a result

$$st_{n-1}(w) \in \left(\{\Theta_\epsilon^{(n-1)}\} \cap B_y(t) \cap \{X_n = y_n\}\right) \subset \left(\{G^{(n-2)}\} \cap B_y(t) \cap \{X_n = y_n\}\right).$$

This implies that $st_{n-1}(w) = y$, since y is a point of local extremum of the coordinate function X_n on $\{G^{(n-2)}\}$ and $\{G^{(n-2)}\} \cap B_y(t) \cap \{X_n = y_n\}$ is finite.

(ii) Result follows immediately from the actual construction of the formula $G^{(i)}$ and the fact that

$$L_i^{i+1}[\omega] \subset \pi_i\big(\{\widehat{G}^{(i-1)}(Y^{(s_{i-1})}, X^{(i+2)}, \omega)\}\big), \qquad 0 < i < n,$$

for any $\omega = (\omega_{i+3}, \dots, \omega_n) \in I^{n-i-2} \subset \mathbb{R}_i^{n-i-2}$.

(iii) In general, if Γ is a curve in $\mathbb{R}_i^k$, $k > m$, and the map ρ is the projection onto the space $\{X_1 = \cdots = X_{k-m} = 0\}$, then the sets $\mathcal{R}_k(\rho(\Gamma)) \setminus \rho(\mathcal{R}_k(\Gamma))$ and $\rho(\mathcal{R}_k(\Gamma)) \setminus \mathcal{R}_k(\rho(\Gamma))$ need not be empty.

Let

$$y = (y_1, \dots, y_n) \in \mathcal{R}_n\big(\pi_{n-1}(\{G^{(n-2)}\})\big),$$

and set $\mathbb{R}_\epsilon = \mathbb{R}_n$. Consider the real closed fields

$$\mathbb{R}_\epsilon \subset \mathbb{R}_{\epsilon,\delta'} \subset \mathbb{R}_{\epsilon,\delta',\delta} \subset \mathbb{R}_{\epsilon,\delta',\delta,\delta''}$$

and elements $\delta' \in \mathbb{R}_{\epsilon,\delta'}$, infinitesimal relative to $\mathbb{R}_\epsilon$, and $\delta'' \in \mathbb{R}_{\epsilon,\delta',\delta,\delta''}$, infinitesimal relative to $\mathbb{R}_{\epsilon,\delta',\delta}$, such that $0 < \delta'' \ll \delta \ll \delta' \ll \varepsilon_{n-1}$.

By the definition of a ramification point, there exist two branches, say G' and G'', of $\pi_{n-1}(\{G^{(n-2)})\}$, such that $y \in G' \cap G''$ and there either exists a pair of points

$$G' \cap \{X_n = y_n - \delta'\}, \quad G'' \cap \{X_n = y_n - \delta'\},$$

or exists a pair of points

$$G' \cap \{X_n = y_n + \delta'\}, \quad G'' \cap \{X_n = y_n + \delta'\}.$$

Let, for definiteness, the first two points exist.

Define

$$y^{(1)} = (y_1^{(1)}, \dots, y_n^{(1)}) = G' \cap \{X_n = y_n - \delta'\},$$

$$y^{(2)} = (y_1^{(2)}, \dots, y_n^{(2)}) = G'' \cap \{X_n = y_n - \delta'\}.$$

Let l be the largest among the numbers $i \in \{1, \dots, n-1\}$ such that $y_i^{(1)} \neq y_i^{(2)}$. Observe that for any α_n with $y_n^{(1)} < \alpha_n < y_n$ and a pair of points $\alpha' = G' \cap \{X_n = \alpha_n\}$, $\alpha'' = G'' \cap \{X_n = \alpha_n\}$, the number l is the largest among i such that $\alpha'_i \neq \alpha''_i$.

Define $\hat{G}'$ and $\hat{G}''$ as shifts of G' and G'' respectively along X_l by δ (i.e., according to the map $X_l \mapsto X_l - \delta$).

For $1 \le l \le n-1$, define the formula $G_{n-l,\delta}^{(n-2)} = G_{n-l}^{(n-2)} \wedge (Z = \delta)$, where

$$G_{n-l}^{(n-2)} = G^{(n-2)}(Y_{s_{n-2}+1}, \dots, Y_{2s_{n-2}}, X_1, \dots, X_{l-1}, X_l - Z, X_{l+1}, \dots, X_n).$$

Obviously, the union $\hat{G}' \cup \hat{G}'' \subset \pi_{n-1}(\{G_{n-l,\delta}^{(n-2)}\})$.

Also define (see Figure 5-4 for an example)

$$\hat{y}^{(1)} = \hat{G}' \cap \{X_n = y_n - \delta'\}, \qquad \hat{y}^{(2)} = \hat{G}'' \cap \{X_n = y_n - \delta'\},$$

$$y^{(3)} = G' \cap \{X_n = y_n - \delta''\}, \qquad y^{(4)} = G'' \cap \{X_n = y_n - \delta''\},$$

$$\hat{y}^{(3)} = \hat{G}' \cap \{X_n = y_n - \delta''\}, \qquad \hat{y}^{(4)} = \hat{G}'' \cap \{X_n = y_n - \delta''\}.$$

Note in particular, that $\hat{y}_j^{(1)} = y_j^{(2)}$ and $\hat{y}_j^{(3)} = y_j^{(4)}$ for $j = l+1, \dots, n$.

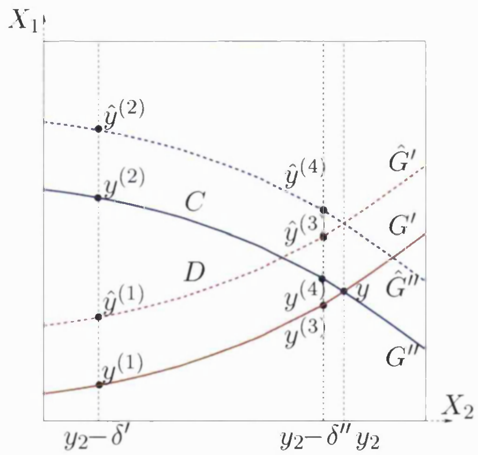


Figure 5-4: The curves $G' \cup G''$ and $\hat{G}' \cup \hat{G}''$ in the vicinity of a ramification point y of $\Gamma^{(n-1)}$ when $n = 2$ and $l = 1$.

Observe that the curve $\pi_{n-1}(\{G^{(n-2)}\})$ is defined over the field $\mathbb{R}_\varepsilon$ (not containing δ', δ'' and δ).

Consider any (definably) connected component G of the curve $\pi_{n-1}(\{G^{(n-2)}\}) \cap \{y_n - \delta' < X_n < y_n - \delta''\}$. Maximal and minimal values of X_n-coordinate on G, if exist, should be elements of $\mathbb{R}_\varepsilon$, thus, for any such value w the distance $|w - y_n|$ is either infinitesimal relative to $\mathbb{R}_{\varepsilon,\delta',\delta,\delta''}$ or $|w - y_n| > a > 0$ for a certain $a \in \mathbb{R}_\varepsilon$. It follows that in the interval $(y_n - \delta', y_n - \delta'')$ there are no maximal or minimal points of X_n-coordinate on G or on the shift $\hat{G}$ of G along X_l by δ.

Now consider the projections $C = \rho_l(G'')$ and $D = \rho_l(\hat{G}')$ on the space $\{X_1 = \cdots = X_{l-1} = 0\}$ equipped with coordinates $X_l, X_{l+1}, \ldots, X_n$. Due to the definition of the formula $G^{(n-2)}$, the connected plane curves C and D are subsets of $\pi_{n-1}(\{G^{(n-2)}\})$ and $\pi_{n-1}(\{G^{(n-2)}_{n-l,\delta}\})$ respectively.

We assume, for definiteness, that $y_l^{(1)} < y_l^{(2)}$. Recall that $\hat{G}' \cup \hat{G}''$ is obtained from $G' \cup G''$ using the δ-shift, and $\delta'' \ll \delta \ll \delta'$. Then the difference $|y_l^{(2)} - y_l^{(1)}|$ is of the order δ' and hence, not infinitesimal relative to $\mathbb{R}_{\varepsilon,\delta',\delta}$, thus $\hat{y}_l^{(1)} < y_l^{(2)}$. Moreover, the difference $|y_l^{(4)} - y_l^{(3)}|$ is of the order δ'' and hence, infinitesimal relative to $\mathbb{R}_{\varepsilon,\delta',\delta}$, thus $\hat{y}_l^{(3)} > y_l^{(4)}$.

Consider the points $a = \rho_l(y^{(2)}), c = \rho_l(y^{(4)}) \in C$ and $b = \rho_l(\hat{y}^{(1)}), d = \rho_l(\hat{y}^{(3)}) \in D$. For any value z_n, with $y_n - \delta' \leq z_n \leq y_n - \delta''$, define the pair of points $z' = C \cap \{X_n = z_n\}$, $z'' = D \cap \{X_n = z_n\}$; then $z'_j = z''_j$ for $l+1 \leq j \leq n$. In particular, for $z_n = y_n - \delta'$ we get that $a_j = b_j$ and for $z_n = y_n - \delta''$ we obtain $c_j = d_j$, for $l+1 \leq j \leq n$. These equalities combined with the two inequalities $a_l > b_l$ and $c_l < d_l$ imply that the curve $C \cup D$ has a ramification point, say $w^{(l)} \in C \cap D$, such that $st_\varepsilon(w_n^{(l)}) = y_n$, where st_ε denotes the standard part relative to the field $\mathbb{R}_\varepsilon = \mathbb{R}_n$ (see Figure 5-5).

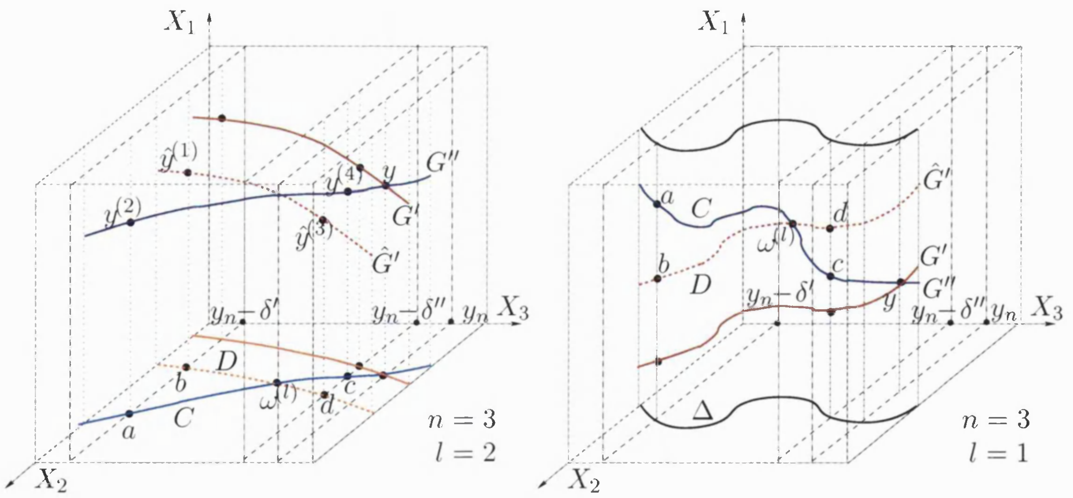


Figure 5-5: Defining a point $\omega^{(l)}$ whose X_n-coordinate is infinitely close to the ramification value y_n of $\Gamma^{(2)}$. Observe that in the figure to the right, the projection of the curve $G' \cup G''$ onto $\{X_1 = 0\}$ coincides with Δ.

Let
$$Q^{(n-1)}_{n-l,\delta} := G^{(n-2)} \wedge G^{(n-2)}_{n-l,\delta}.$$
Since $\pi_{n-1}(\{Q^{(n-1)}_{n-l,\delta}\}) = \pi_{n-1}(\{G^{(n-2)}\}) \cap \pi_{n-1}(\{G^{(n-2)}_{n-l,\delta}\})$, we deduce that

$$w^{(l)} \in \rho_l(\pi_{n-1}(\{Q^{(n-1)}_{n-l,\delta}\})) \subset \pi_{n-1}(\{Q^{(n-1)}_{n-l,\delta}\}).$$

It follows from the actual construction of the formula $\Theta^{(n-1)}_r(Y^{(2s_{n-2}}, X)$ that $y^{(l)} = st_\varepsilon(w^{(l)}) \in \pi_{n-1}(\{\Theta^{(n-1)}_r\})$ and that $\rho_n(y^{(l)}) = y_n$. □

Remark 5.1.3. *For index i, $0 \leq i \leq n-1$, and any $\omega \in I^{n-i-1}$, suppose that*

$$y = (y_1, \dots, y_i, y_{i+1}, \omega) \in \pi_i\big(\{\Theta^{(i)}(Y^{(2s_{i-1})}, X^{(i+1)}, \omega)\}\big) \subset \mathbb{R}^{i+1}_i.$$

In what follows, we assume that in the description of the cell decomposition $\mathcal{D}$ of the unit cube I^n presented in Section §4.2, for each i, we add the point y to the set $\Omega^{(i)}_s[\omega]$ (see Corollary 4.3.5). By Lemma 5.1.1, there is a finite number of such points.

Lemma 5.1.4. *For index i, $0 \leq i \leq n-1$, and any $\omega = (w_{i+2}, \dots, w_n) \in I^{n-i-1}$,*

$$\Omega^{(i)}[\omega] = \pi_i\big(\{\widehat{G}^{(i)}(Y^{(s_i)}, X^{(i+1)}, \omega)\}\big).$$

in the space $\mathbb{R}^{i+1}_{i+1}$ identified with $\{X_{i+2} = \omega_{i+2}, \dots, X_n = \omega_n\}$.

Proof. We shall proceed by induction on i.

First consider the base of the induction for $i = 0$. Fix variables $X_2, \dots, X_n$. If $f(X)$ is identically zero, then $\{f^{(0)}(X) - \delta = 0\} = \emptyset$. In this case,

$$\{\widehat{G}^{(0)}\} = \{0, \varepsilon_0, 1 - \varepsilon_0, 1\} = \Omega^{(0)} \subset \mathbb{R}_1.$$

If $f(X)$ defines a finite collection of points then $H^{(0)}_\delta$ also defines a finite set of points; moreover, every point from $\{f^{(0)}(X) = 0\}$ is infinitesimally close to a point in this set. It follows from Lemma A.3 and Lemma A.8, that

$$\{\Theta^{(0)}_e\} = st_1\big(\{f^{(0)}(X) - \delta = 0\}\big) \cup \{0, 1\} = \{X_1(X_1 - 1) f^{(0)}(X) = 0\}.$$

Thus,
$$\{\widehat{G}^{(0)}\} = \{\widehat{G}^{(0)}_0\} = \{\Theta^{(0)}_e\} \cap \{0 \leq X_1 \leq 1\} = \Omega^{(0)}_0 \subset \mathbb{R}.$$

Suppose that the lemma is proved for all $i < n-1$. We now consider the case

$i = n-1$. By inductive hypothesis, for every fixed value ω_n of X_n-coordinate we have

$$\Omega^{(n-2)}[\omega] = \pi_{n-2}\big(\{\widehat{G}^{(n-2)}(Y^{(s_{n-2})}, X^{(n-1)}, \omega_n)\}\big) \subset \mathbb{R}^{n-1}_{n-1}.$$

Thus, in the space $\mathbb{R}^n_{n-1}$ equipped with coordinates $X_1, \dots, X_n$, the curve

$$\widehat{\Gamma}^{(n-1)} = \pi_{n-1}(\{\widehat{G}^{(n-2)}\}).$$

As a result the closed curve $\Gamma^{(n-1)} = \pi_{n-1}(\{G^{(n-2)}\})$ and the set of its special points relative to X_n-coordinate

$$\Omega_s^{(n-1)} = \mathcal{S}_n\big(\pi_{n-1}(\{G^{(n-2)}\})\big).$$

Hence, $\Omega_s^{(n-1)} = \mathcal{E}_n\big(\pi_{n-1}(\{G^{(n-2)}\})\big) \cup \mathcal{R}_n\big(\pi_{n-1}(\{G^{(n-2)}\})\big)$. It follows directly from Lemma 5.1.2 (i),(iii) and Remark 5.1.3, that

$$\rho_n\big(\Omega_s^{(n-1)}\big) = \rho_n\big(\pi_{n-1}\{\Theta^{(n-1)}\}\big), \tag{5.1}$$

that is, for every special point of the curve $\Gamma^{(n-1)}$, there is a point in the finite set $\{\Theta^{(n-1)}\}$ having the same X_n-coordinate.

We first prove the inclusion $\Omega^{(n-1)} \subset \pi_{n-1}(\{\widehat{G}^{(n-1)}\})$. Let $y \in \Omega^{(n-1)}$. By the definition of $\Omega^{(n-1)}$, the point y belongs to one of the intersections $\widehat{\Gamma}^{(n-1)} \cap \{X_n = z_n\}$ or $\widehat{\Gamma}^{(n-1)} \cap \{X_n = z_n + \varepsilon_{n-1}\}$, where $z = (z_1, \dots, z_n)$ is either a special point of $\Gamma^{(n-1)}$, or (by Remark 5.1.3) a point in $\pi_{n-1}(\{\Theta^{(n-1)}\})$.

In what follows, unless otherwise stated, we assume that the curve $\widehat{\Gamma}^{(n-1)}$ belongs to the space $\mathbb{R}^n_{n-1}$ equipped with coordinates $Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n$. The formula $\widehat{G}^{(n-2)}$ stands for $\widehat{G}^{(n-2)}(Y^{(s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n)$ and π_{n-1} is the projection on $Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n$. Consider the following cases.

Case 1 Let $y \in \widehat{\Gamma}^{(n-1)} \cap \{X_n = z_n\}$, where either $z = (z_1, \dots, z_n) \in \Omega_s^{(n-1)}$, or

$$z \in \pi_{n-1}(\{\Theta^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n)\}).$$

Case 1.1 Let $z \in \Omega_s^{(n-1)}$. Then by (5.1), there exists a point

$$u \in \{\Theta^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n)\} \cap \{X_n = z_n\}.$$

According to the formula,

$$\widehat{G}_0^{(n-1)}(Y^{(s_{n-1})}, X^{(n)}) := \widehat{G}^{(n-2)}(Y_{1+2s_{n-2}}, \dots, Y_{3s_{n-2}}, X) \wedge$$
$$\Theta^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n).$$

It follows from Lemma 5.1.1, that

$$L = \{\widehat{G}^{(n-2)}(Y_{1+2s_{n-2}}, \dots, Y_{3s_{n-2}}, X)\}$$

is a $\mathcal{F}$-semianalytic curve in the $(s_{n-2}+n)$-dimensional space equipped with coordinates $Y_{1+2s_{n-2}}, \dots, Y_{3s_{n-2}}, X_1, \dots, X_n$, whose intersection with the hyperplane $\{X_n = \omega_n\}$, for each fixed value ω_n of X_n-coordinate, consists of a finite number of points. Then for $\omega_n = z_n$, there exists a point

$$p \in (L \cap \{X_n = z_n\}) \subset \{\widehat{G}_0^{(n-1)}(Y^{(s_{n-1})}, X)\} \subset \{\widehat{G}^{(n-1)}(Y^{(s_{n-1})}, X)\},$$

such that

$$\pi_{n-1}(p) = y \in \Omega^{(n-1)} \subset \widehat{\Gamma}^{(n-1)},$$

where this time, $\widehat{\Gamma}^{(n-1)} = \pi_{n-1}(L)$ is a curve in the space $\mathbb{R}^n_{n-1}$ equipped with coordinates $X_1, \dots, X_n$, and π_{n-1} is the projection on this space. In particular, $\Omega_0^{(n-1)} \subset \pi_{n-1}(\{\widehat{G}_0^{(n-1)}\})$.

Case 1.2 Let $z' \in \{\Theta^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n)\}$ so that $\pi_{n-1}(z') = z$. Then this case can be proved in exactly the same way as Case 1.1, by taking $u = z'$.

Case 2 Let $y \in \widehat{\Gamma}^{(n-1)} \cap \{X_n = z_n \pm \varepsilon_{n-1}\}$, where either $z = (z_1, \dots, z_n) \in \Omega_s^{(n-1)}$, or

$$z \in \pi_{n-1}(\{\Theta^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n)\}).$$

Then this case can be proved, by employing the same arguments as in Case 1, with the formula $\widehat{G}_1^{(n-1)}$ replacing $\widehat{G}_0^{(n-1)}$.

We have so far proved that $\Omega^{(n-1)} \subset \pi_{n-1}(\{\widehat{G}^{(n-1)}\})$. The inclusion

$$\pi_{n-1}(\{\widehat{G}^{(n-1)}\}) \subset \Omega^{(n-1)}$$

and in particular, the inclusion $\pi_{n-1}(\{\widehat{G}_0^{(n-1)}\}) \subset \Omega_0^{(n-1)}$, can be established in a similar way. □

Remark 5.1.5. *The assumption made in Remark 5.1.3 is crucial for the proof of Lemma 5.1.4. By Lemma 5.1.2, for any vector $\omega \in I^{n-i-2}$, the following inclusion*

$$\rho_{i+2}\left(\mathcal{S}_{i+2}(\pi_{i+1}(\{G^{(i)}(Y^{(s_i)}, X^{(i+2)}, \omega)\}))\right) \subset \rho_{i+2}\left(\pi_{i+1}(\{\Theta^{(i+1)}(Y^{(2s_i)}, X^{(i+2)}, \omega)\})\right)$$

is valid, but unfortunately the reverse inclusion may not hold. Suppose we do not make *this assumption; furthermore suppose that the strict inclusion*

$$\Omega^{(i)}[\omega_{i+2}, \omega] \subset \pi_i(\{\widehat{G}^{(i)}(Y^{(s_i)}, X^{(i+1)}, \omega_{i+2}, \omega)\})$$

holds for some i, $1 \leq i \leq n-2$ and any $\omega_{i+2} \in I$, $\omega \in I^{n-i-2}$. Then clearly,

$$\Gamma^{(i+1)}[\omega] \subset \pi_{i+1}\big(\{G^{(i)}(Y^{(s_i)}, X^{(i+2)}, \omega)\}\big).$$

Even in this case $\mathcal{R}_{i+2}(\Gamma^{(i+1)}[\omega]) \subset \mathcal{R}_{i+2}\big(\pi_{i+1}(\{G^{(i)}(Y^{(s_i)}, X^{(i+2)}, \omega)\})\big)$, but in general the set

$$\mathcal{E}_{i+2}(\Gamma^{(i+1)}[\omega]) \setminus \mathcal{E}_{i+2}\big(\pi_{i+1}(\{G^{(i)}(Y^{(s_i)}, X^{(i+2)}, \omega)\})\big)$$

may not be empty – for example, an isolated point of the curve $\Gamma^{(i+1)}[\omega]$, may not be an isolated point of the curve $\pi_{i+1}\big(\{G^{(i)}(Y^{(s_i)}, X^{(i+2)}, \omega)\}\big)$.

The descending phase of our method writes out formulas $\widehat{G}^{(k)}$ for all $0 \leq k \leq n-1$, using the above recursive definitions.

5.2 Defining the cylindrical cell decomposition

Having constructed, for any i, $0 \leq i \leq n-1$, and $\omega \in I^{n-i-1}$, an existential $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(i+1)}$-formula determining the set $\Omega^{(i)}[\omega]$, we show next how to define the cylindrical cell decomposition $\mathcal{D}$ of the unit cube I^n, described in Section §4.2, in such a way that each cell in $\mathcal{D}$ is a $\mathcal{F}$-subanalytic set. This is the purpose of the ascending phase of our method. It consists of the following recursive procedure.

For any index i, $0 \leq i \leq n-1$, define the $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(i)}$-formula

$$\Phi_L^{(i)}(X) := \exists Y^{(s_i)}\big(\widehat{G}_0^{(i)}(Y^{(s_i)}, X) \wedge (X_1 = \cdots = X_i = 0)\big).$$

For any vector $\omega = (\omega_{i+2}, \ldots, \omega_n) \in I^{n-i-1}$, let

$$\Omega_L^{(i)}[\omega] = \{\Phi_L^{(i)}(X^{(i+1)}, \omega)\} \subset \mathbb{R}_i^{i+1}.$$

Notice that $\Omega_L^{(i)}[\omega] = \Omega_0^{(i)}[\omega] \cap L_i^n(0)[\omega]$ is precisely the set obtained as the projection of special points $\Omega_s^{(i)}[\omega]$ of the parametric curve $\Gamma^{(i)}[\omega] \subset \mathbb{R}_i^{i+1}$ onto the subspace

$$\{X_1 = \cdots = X_i = 0, X_{i+2} = \omega_{i+2}, \ldots, X_n = \omega_n\}.$$

We describe now, how to define for each value of $i \in \{0, \ldots, n\}$, every cell $C_\alpha^{(i)}$, with index $\alpha = (0, \ldots, 0, \alpha_{i+1} \ldots, \alpha_n)$, in a cylindrical cell decomposition $\mathcal{D}^{(i)}$ of $L_i^n(0)$

(which is compatible with the projection of V onto the subset $L_i^n(0)$ of $\mathbb{R}_n^{n-i}$ equipped with coordinates $X_{i+1}, \dots, X_n$).

For each k, $0 \le k \le n$, define $i = n - k$.

For $k = 0$, set $\alpha = (0, \dots, 0)$ and $C_\alpha^{(n)} = \{(0, \dots, 0)\}$ to be the only cell of the cylindrical cell decomposition $\mathcal{D}^{(n)}$ of $L_n^n(0)$.

Suppose that at step k, a cylindrical cell decomposition $\mathcal{D}^{(n-k)}$ of the cube $L_{n-k}^n(0)$ has been defined.

At the next step $k+1$, with $i = n - k - 1$, the input is the cylindrical cell decomposition $\mathcal{D}^{(i+1)}$ of the cube $L_{i+1}^n(0)$ obtained at the previous step. For each cell $C_\alpha^{(i+1)} \in \mathcal{D}^{(i+1)}$, denote by $Z(C_\alpha^{(i+1)})$ the set $C_\alpha^{(i+1)} \times [0,1]$, which is the bounded cylinder over $C_\alpha^{(i+1)}$ and along X_{i+1}, contained in the cube $L_i^n(0)$. We construct a cell decomposition of $Z(C_\alpha^{(i+1)})$. Observe that for any point $p = (p_{i+2}, \dots, p_n) \in C_\alpha^{(i+1)}$, the cardinality of the set $\Omega_L^{(i)}[p]$ is finite and constant over $C_\alpha^{(i+1)}$. The o-minimality of the structure $\widetilde{\mathbb{R}}_{An}^{(i)}$ (which follows from the o-minimality of the expansion $\mathbb{R}_{An}$ of the real ordered field by restricted analytic functions) implies that this number does not exceed some natural number K_α.

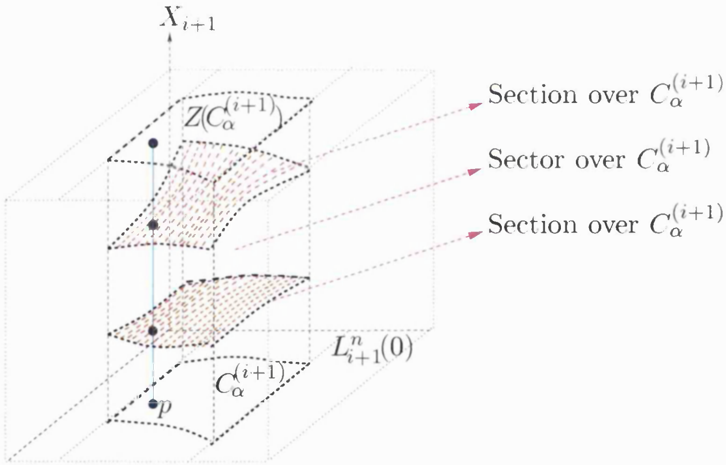


Figure 5-6: Defining the cylindrical cells inductively.

Introduce new variables $Y_{j,1}^{(i)}, \dots, Y_{j,s_i}^{(i)}, X_{j,i+1}^{(i)}$, $1 \le j \le K_\alpha$, and denote

$$X_j^{(i)} = (0, \dots, 0, X_{j,i+1}^{(i)}, p).$$

We now describe all cells in $Z(C_\alpha^{(i+1)}) \subset L_i^n(0)$ by the following formulas.

- Sections over $C_\alpha^{(i+1)}$: for $1 \le j \le K_\alpha$

$$C_{\alpha'}^{(i)} = \Big\{ (0,\dots,0,W_{i+1},\dots,W_n) \in Z(C_\alpha^{(i+1)}) \;:\; (\exists X_{1,i+1}^{(i)}) \cdots (\exists X_{K_\alpha,i+1}^{(i)})$$

$$\Big(\bigwedge_{1\le j\le K_\alpha} \Phi_L^{(i)}(X_j^{(i)}) \;\wedge\; X_{1,i+1}^{(i)} < \cdots < X_{j,i+1}^{(i)} = W_{i+1} < X_{j+1,i+1}^{(i)} < \cdots < X_{K_\alpha,i+1}^{(i)}\Big)\Big\},$$

where the index $\alpha' = (0,\dots,0,2j-2,\alpha_{i+2},\dots,\alpha_n)$.

- Sectors over $C_\alpha^{(i+1)}$: for $1 \le j \le K_\alpha - 1$

$$C_{\alpha'}^{(i)} = \Big\{ (0,\dots,0,W_{i+1},\dots,W_n) \in Z(C_\alpha^{(i+1)}) \;:\; (\exists X_{1,i+1}^{(i)}) \cdots (\exists X_{K_\alpha,i+1}^{(i)})$$

$$\Big(\bigwedge_{1\le j\le K_\alpha} \Phi_L^{(i)}(X_j^{(i)}) \;\wedge\; X_{1,i+1}^{(i)} < \cdots < X_{j,i+1}^{(i)} < W_{i+1} < X_{j+1,i+1}^{(i)} < \cdots < X_{K_\alpha,i+1}^{(i)}\Big)\Big\},$$

where the index $\alpha' = (0,\dots,0,2j-1,\alpha_{i+2},\dots,\alpha_n)$.

Combining the cell decompositions of $Z(C_\alpha^{(i+1)})$ for all cells $C_\alpha^{(i+1)}$ in $\mathcal{D}^{(i+1)}$, we obtain a cylindrical cell decomposition $\mathcal{D}^{(i)}$ of the cube $L_i^n(0)$. This concludes the description of the last phase of our method.

Given any $\widetilde{\mathcal{L}}_{\mathcal{F}}$-formula Φ, we can apply the above construction to the set $S \subset \mathbb{R}_n^n$ defined by the quantifier-free part Θ of Φ, to obtain existential $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(n)}$-formulas defining a family of cells that constitutes a cylindrical decomposition of the unit cube $I^n \subset \mathbb{R}_n^n$ compatible with S. Precisely because of the cylindrical arrangement of the cells, by treating existential and universal quantifiers in Φ as disjunctions and conjuctions of existential formulas defining lower dimensional cells over which Φ is true, we are able to construct an existential $\widetilde{\mathcal{L}}_{\mathcal{F}}^{(n)}$-formula Ψ such that

$$\mathbb{R}_n \models \forall X_1 \cdots \forall X_n \big(\Phi(X_1,\dots,X_n) \longleftrightarrow \Psi(X_1,\dots,X_n)\big).$$

According to the construction of Ψ, the only elements of $\mathbb{R}_n \setminus \mathbb{R}$ which appear in this formula as constants are the infinitesimals $\varepsilon_0,\dots,\varepsilon_{n-1}$. Denote by Ψ' the $\widetilde{\mathcal{L}}_{\mathcal{F}}$-formula obtained by replacing in Ψ these infinitesimal elements by some new variables $Z_1,\dots,Z_n$ respectively. Then clearly,

$$\mathbb{R}_n \models \forall X_1 \cdots \forall X_n \big(\Phi(X_1,\dots,X_n) \longleftrightarrow \exists Z_1 \cdots \exists Z_n\, \Psi'(X_1,\dots,X_n)\big).$$

Since the fields $\mathbb{R}$ and $\mathbb{R}_n$ are elementarily equivalent, it follows that the expansion $\mathcal{R}_{\mathcal{F}}$ of the real ordered field by restricted analytic functions from $\mathcal{F}$ is model complete (Theorem 3.2.6). This implies in particular, that the complement (within the unit cube) of a $\mathcal{F}$-subanalytic subset of $\mathbb{R}^n$ is $\mathcal{F}$-subanalytic.

Chapter 6

Application to restricted Pfaffian functions

As an immediate consequence of our work so far, when the functions in $\mathcal{F}$ are Pfaffian, we get that the complement (within the unit cube) of a sub-Pfaffian set is sub-Pfaffian [Wil96, Gab96]. In fact, it is possible to modify the methods we have employed in the proof of the complement theorem in Chapter 5, by taking into consideration effective bounds on finite properties of this important class of analytic functions, given in [Gab98] and [Kho91], to obtain an (conditional) algorithm for producing cylindrical cell decompositions of restricted sub-Pfaffian sets with the best up-to-date complexity bounds. Our algorithm in particular, is able to compute complements of restricted sub-Pfaffian sets. This is the subject of Section §6.1. In the remaining part of this Chapter we discuss an alternative recursive construction which allows us to prove a doubly exponential in the number of variables bound on the number of cells in cylindrical cell decompositions of sub-Pfaffian sets. This is an improvement in relation to the analogous bound which follows from the complexity analysis of either the algorithm presented in Section §6.1, or the one appearing in [GV01]. The o-minimality of the expansion $\mathcal{R}_{RP}$ of the real ordered field by restricted Pfaffian functions implies that the same upper bound is also valid for the number of connected components of $\mathcal{R}_{RP}$-definable sets. A single exponential in the number of variables upper bound for the sum of Betti numbers (and so in particular for the number of connected components) of subsets of $\mathbb{R}^n$ defined by expressions with no negations, involving atomic formulas of the kind $f \geq 0$, where f is a restricted Pfaffian function, appeared recently in [GVZ02].

6.1 An algorithm for computing complements of restricted sub-Pfaffian sets

Let S be a semi-Pfaffian set in some neighbourhood G of the closed unit cube I^n of $\mathbb{R}^n$ with Pfaffian chain F and $W = \rho_m(S) \subset I^{n-m+1} \subset \mathbb{R}^{n-m+1}$, $1 \le m \le n$, where ρ_m denotes the projection map omiting the first $m-1$ coordinates. Theorem 3.2.5 states that the complement $\widetilde{W}$ of W within the cube I^{n-m+1} is sub-Pfaffian and more precisely, that it can be described by functions from the algebra generated by the members of F, their partial derivatives, constants 0 and 1, and coordinate functions. In this section we construct an (conditional) algorithm for *computing* the complement $\widetilde{W}$, by incorporating into our method, results regarding some finite properties of semi-Pfaffian sets. We employ Khovanskii's bound on the number of definably connected components of a semi-Pfaffian set (Proposition 3.3.13), and Gabrielov's algorithm for computing the closure of a restricted semi-Pfaffian set (Lemma 3.3.16).

As a model of computation we use a version of a *real numbers machine* (Blum-Shub-Smale model, see [BCSS97] and the survey paper [MM97]) equipped with an *oracle* for deciding the feasibility of any system of Pfaffian equations and inequalities. An oracle is a subroutine which can be used by the algorithm any time the latter needs to check feasibility. We assume that this procedure always gives the correct answer though we do not specify how it actually works. For some classes of Pfaffian functions the feasibility problem is decidable on real numbers machines or Turing machines with explicit (singly-exponential) complexity bounds. Apart from polynomials, such class form, for example, terms of the kind $P(e^h, X_1, \dots, X_n)$ where h is a fixed polynomial in $X_1, \dots, X_n$ and P is an arbitrary polynomial in $X_0, X_1, \dots, X_n$ (see [Vor92]). For such classes the oracle can be replaced by a deciding procedure, and we get an algorithm in the usual sense. As far as the computational complexity is concerned, we assume that each oracle call has a unit cost.

The main result that we prove in this section is the following.

Theorem 6.1.1. *There exists an algorithm which produces a cylindrical cell decomposition $\mathcal{D}$ of the unit cube $I^n \subset \mathbb{R}^n_n$, compatible with a semi-Pfaffian set S having format (N, α, β, r, n), and consisting of*

$$(\alpha + \beta N)^{r^{O(n)} 2^{O(n^2)}}$$

cells, provided we are given an oracle for deciding the feasibility of any system of Pfaffian equations and inequalities. Assuming that each oracle call has a unit cost, each *cell is a sub-Pfaffian set of the format*

$$\left((\alpha+\beta N)^{r^{O(n)}2^{O(n^2)}},\ \alpha,\ (\alpha+\beta N)^{r^{O(n)}2^{O(n^2)}},\ (\alpha+\beta N)^{r^{O(n)}2^{O(n^2)}},\ (\alpha+\beta N)^{r^{O(n)}2^{O(n^2)}}\right).$$

The complexity of the algorithm does not exceed

$$(\alpha+\beta N)^{r^{O(n)}2^{O(n^2)}}.$$

In particular, with the help of the oracle, the algorithm can eliminate one sort of quantifier (either $\exists$ or $\forall$) from first-order formulas involving restricted Pfaffian functions.

The algorithm recursively constructs the decomposition $\mathcal{D}$ described in Section §4.2, when the input semianalytic set S is actually semi-Pfaffian with format (N, α, β, r, n). It consists of two phases: the descending phase and the ascending phase.

6.1.1 The descending phase

It follows from the definition of the single Pfaffian function f (page 56, equation (4.1)) that the semi-Pfaffian set $V = (\{f = 0\} \cap I^n) \cup I_1^n$ has format $(O(n2^n), \alpha, 2\beta N, r, n+1)$. Denote $N_V = O(n2^n)$ and $D_V = O(\alpha + \beta N)$.

Lemma 6.1.2. *For each k, $0 \le k \le n-1$, the format of the set defined by $\widehat{G}^{(k)}$ is $(N_k, \alpha, \beta_k, r_k, m_k)$, where*

$$N_k = (\alpha+\beta N)^{(r+n)^{O(k)}2^{O(k^2)}}, \beta_k = (\alpha+\beta N)^{(r+n)^{O(k)}2^{O(k^2)}}, r_k = r3^k, m_k = O(n3^k).$$

Proof. At step k, $0 \le k \le n-1$, let $m_k = s_k + n$ be the total number of variables and r_k the size of the Pfaffian chain for the functions in $\widehat{G}^{(k)}$. Recall that $s_0 = 0$ and $s_k = 3s_{k-1} + k$. It can be shown that $s_k = \sum_{0 \le j \le k-1} 3^j(k-j)$. Then $m_k = n + \sum_{0 \le j \le k-1} 3^j(k-j) = O(n + k3^k) = O(n3^k)$.

For $k = 0$, the order $r_0 = r$ since the operation of taking closure (see Lemma 3.3.16) leaves the order of the Pfaffian chain unchanged. Suppose that $F(Y^{(s_{k-1})}, X) = \left(f_1(Y^{(s_{k-1})}, X), \dots, f_{r_{k-1}}(Y^{(s_{k-1})}, X)\right)$ is the Pfaffian chain of the set defined by $\widehat{G}^{(k-1)}$. Notice that the order of the Pfaffian chain of the set defined by $Q_{j,Z}^{(k)}$, $1 \le j \le k$, is $2r_{k-1}$ since we need to add in this chain the same functions as before, but with the variables $Y^{(s_{k-1})}, X_{k+1-j}$ replaced in this order, by $Y_{1+s_{k-1}}, \dots, Y_{2s_{k-1}}, X_{k+1-j} - Z$. Thus the size of the Pfaffian chain of the set defined by $Q_Z^{(k)}$ is equal to $(k+1)r_{k-1}$. According to Lemma 3.3.16, there is a Boolean formula, say $Q^{(k)}$ with atomic Pfaffian functions in variables $Y^{(2s_{k-1})}, X_1, \dots, X_n, Z$, having the same common Pfaffian chain as $Q_Z^{(k)}$, such that $\{Q^{(k)}\} = cl\{Q_Z^{(k)} \wedge (Z > 0)\}$. The formula $\Theta_r^{(k)} := Q^{(k)} \wedge (Z = 0)$ is equivalent to an expression involving Pfaffian functions only in variables $Y^{(2s_{k-1})}, X$. By substituting the value 0 for the variable Z in every function present in the Pfaffian chain of the set $Q^{(k)}$, we deduce that the Pfaffian chain of the set defined by $\Theta_r^{(k)}$ is $F(Y^{(s_{k-1})}, X), F(Y_{1+s_{k-1}}, \dots, Y_{2s_{k-1}}, X)$. Similarly, the formula $\Theta_e^{(k)}$ is equivalent to an expression involving Pfaffian functions only in variables $Y^{(2s_{k-1})}, X$ having common Pfaffian chain $F(Y^{(s_{k-1})}, X)$. The Pfaffian chain of the set defined by $\widehat{G}^{(k)}$ is

$$F(Y_{1+2s_{k-1}}, \dots, Y_{3s_{k-1}}, X)\ ,\ F(Y^{(s_{k-1})}, Y_{1+3s_{k-1}}, Y_{k+3s_{k-1}}, X_{k+1}, \dots, X_n)$$

$$F(Y_{1+s_{k-1}}, \dots, Y_{2s_{k-1}}, Y_{1+3s_{k-1}}, Y_{k+3s_{k-1}}, X_{k+1}, \dots, X_n).$$

We conclude that the order of the set defined by $\widehat{G}^{(k)}$ is $r_k = 3r_{k-1} = r3^k$.

Let

$$p_k = \prod_{0 \le j \le k} m_j = n^{k-1} 2^{O(k^2)}, \qquad q_k = \prod_{0 \le j \le k} (m_j + r_j) = (r+n)^{k-1} 2^{O(k^2)}.$$

For $k = 0$, applying the bounds for Closure Algorithm, stated in Lemma 3.3.16, we get

$$N_0 = (2^{r^2} N_V)^{O(m_0(m_0+r_0))} (m_0 D_V)^{O(m_0(m_0+r_0)^2)}$$

and

$$\beta_0 = 2^{m_0 r_0^2} (m_0 D_V)^{O(m_0(m_0+r_0))}$$

Note that at each other step we perform two iterations of the closure operation. It can be seen that

$$\beta_k = (2^r p_k D_V)^{O(p_k^2 q_k^2)} = D_V^{(r+n)^{O(k)} 2^{O(k^2)}}$$

and

$$N_k = (2^{r^2} p_k N_V \beta_k^{2k})^{O(p_k^2 q_k^3)} = D_V^{(r+n)^{O(k)} 2^{O(k^2)}}$$

□

In the descending phase, the algorithm writes out formulas $\widehat{G}^{(k)}$, for all k, $0 \le k \le n-1$, as described in Section §5.1. It follows from Lemma 6.1.2, that the complexity of this phase of the algorithm does not exceed

$$(\alpha + \beta N)^{r^{O(n)} 2^{O(n^2)}}.$$

6.1.2 The ascending phase

The ascending phase of the algorithm follows the same recursive procedure that was presented in Section §5.2, in order to define all cells in the cylindrical cell decomposition

$\mathcal{D}$. But in addition, by employing Khovanskii's bound on the number of definably connected components of a semi-Pfaffian set (see Proposition 3.3.13), in conjuction with an oracle for deciding concistency of systems of Pfaffian equations and inequalities, we are able to determine the exact number of points in the finite (parametric) set $\Omega^{(i)}$.

On each recursion step k, $1 \leq k \leq n$ (with $i = n - k$) of the ascending phase, Corollary 3.3.14 implies that for any point $p = (p_{n-k+2}, \dots, p_n) \in C_\alpha^{n-k+1} \subset I^{k-1}$ the cardinality of the set

$$\Omega_L^{(n-k)}[p] = \{\Phi_L^{(n-k)}(X_1, \dots, X_{n-k+1}, p)\},$$

does not exceed

$$M_{n-k} = (\alpha + \beta N)^{(r+n)^{O(n-k)} 2^{O((n-k)^2)}}.$$

The algorithm finds the exact number K_α of distinct points in $\Omega_L^{(i)}[p]$ by testing, using the oracle, for each m, $1 \leq m \leq M_i$, whether the statement

$$(\exists X_1^{(i)}) \cdots (\exists X_m^{(i)}) \Big[\Big(\bigwedge_{1 \leq j \leq m} \Phi_L^{(i)}(X_j^{(i)}) \Big) \wedge \Big(\bigwedge_{1 \leq r \leq m-1} \bigwedge_{1 \leq j \leq r} (X_{r+1,i+1}^{(i)} \neq X_{j,i+1}^{(i)}) \Big) \Big]$$

is true. K_α is the maximal m for which the statement holds.

The degrees of the Pfaffian functions involved in the defining formula of any cell in $\mathcal{D}^{(n-k)}$ remain the same as in $\widehat{G}^{(n-k)}$, while the numbers of variables, orders and atomic formulas in such a defining formula is multiplied by at most $M_{n-k} + 1$. The number of cells is increased on a recursion step according to the formula $|\mathcal{D}^{(n-k)}| \leq O(M_{n-k})|\mathcal{D}^{(n-k+1)}|$ and $|\mathcal{D}^{(n)}| = 1$, so $|\mathcal{D}^{(0)}| \leq O(M_n)^n$.

Putting together these two phases we establish Theorem 6.1.1.

6.2 Obtaining an improved upper bound on the number of cylindrical cells

In the remaining part of this chapter we discuss a method which yields a doubly exponential in the number of variables upper bound on the number of cells of the cylindrical cell decomposition $\mathcal{D}$ described in Section §4.2.

The idea is to construct an auxiliary finite sub-Pfaffian set, having the property that each point from $\Omega^{(n-1)}$ is infinitely close to some of its members. As a result, the cardinality of this set, which we can estimate using Khovanskii's bound (see Corollary 3.3.14) exceeds the number of points of $\Omega^{(n-1)}$; a similar estimate then follows for the number of cells in $\mathcal{D}$, since $|\mathcal{D}| = O(2^n|\Omega^{(n-1)}|)$. The proof of some upper bound on the number of cells in cylindrical cell decompositions of semi-Pfaffian sets, implies in particular that the same upper bound also applies for the number of connected components of sets definable by first-order formulas involving Pfaffian functions.

We emphasize that here we are only interested in obtaining an estimate for the number of cells in $\mathcal{D}$; it is not our intention to actually define the cells in this decomposition. In the process of constructing "approximating" sets to $\Omega^{(i)}$, $0 \leq i \leq n-1$, we only use systems of Pfaffian equations, no Pfaffian inequalities are involved. As it is explained later, an extra degree of difficulty arises precisely because we do not "pass to limit" at each step of this inductive construction but instead we keep working with the "approximating" sets through out. It is worth pointing out though that during this process, neither Gabrielov's algorithm for computing the closure of a restricted semi-Pfafian set, nor the *oracle* for deciding emptiness of semi-Pfaffian sets (both of which were essential for the construction of the algorithm presented in Section §6.1) are necessary.

At some point of the following construction it would be convenient to have at our disposal one more simple definition which follows easily from the description of the cylindrical cell decomposition $\mathcal{D}$ of the unit cube compatible with a given semi-Pfaffian set (presented in Section §4.2). By applying the inductive hypothesis to the cube $I^n[\omega_{i+2}, \omega] \subset \mathbb{R}^{i+1}_{i+1}$, for any value $\omega_{i+2} \in I$ and any vector $\omega = (\omega_{i+3}, \dots, \omega_n) \in I^{n-i-2}$, $0 \leq i \leq n-2$, a parametric curve $\Gamma^{(i)}[\omega_{i+2}, \omega]$ can be determined. Define

$$\Omega^{(i)}_e[\omega_{i+2}, \omega] := \mathcal{E}_{i+1}\big(\Gamma^{(i)}[\omega_{i+2}, \omega]\big)$$

to be the points of local extremum of the coordinate function X_{i+1} on $\Gamma^{(i)}[\omega_{i+2}, \omega]$ and denote

$$\widehat{\Gamma}^{(i+1)}_e[\omega] := \bigcup_{\omega_{i+2} \in [0,1]} \Omega^{(i)}_e[\omega_{i+2}, \omega], \qquad \Gamma^{(i+1)}_e[\omega] := cl\big(\widehat{\Gamma}^{(i+1)}_e[\omega]\big).$$

We now formulate the main results of this section.

Theorem 6.2.1. *Let $f : G \longrightarrow \mathbb{R}$ be a Pfaffian function of the order not exceeding r and degree (α, β) in an open domain $G \subset \mathbb{R}^n$ containing the unit cube I^n. Then there exists a cylindrical cell decomposition of I^n compatible with $Zer(f)$, with the total number of cells less than*

$$2^{3^{2n-2}r^2}(n!(\alpha+\beta))^{O(3^{n-1}(r+n))}. \tag{6.1}$$

As a result, the number of connected components of any set that can be defined by finitely many applications of union, intersection, complementation and projection operations starting from $Zer(f)$ does not exceed the above bound.

Using the same trick as in Section §4.1, we are able to obtain the following generalization.

Theorem 6.2.2. *Consider a $\mathcal{L}_{RP}$-formula $\Phi(X_1, \dots, X_k, Y_1, \dots, Y_m)$ in prenex form with quantifier-free part*

$$\Theta := \bigvee_{1 \le i \le M} (f_i = 0, g_{i,1} > 0, \dots, g_{i,J_i} > 0),$$

where f_i, g_{ij} are restricted Pfaffian functions in $n \ge k + m$ variables with a common Pfaffian chain of the order not exceeding r and degrees (α, β). Then the number of connected components of the set

$$\{(\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k : \mathbb{R}_{RP} \models \Phi(\alpha_1, \dots, \alpha_k, c_1, \dots, c_m)\}$$

for any choice of $(c_1, \dots, c_m) \in \mathbb{R}^m$ is at most

$$2^{3^{2n} r^2} (n!(\alpha + 2N\beta))^{O(3^n (r+n))}, \tag{6.2}$$

where $N = 1 + \sum_{1 \le i \le M} (J_i + 1)$.

6.2.1 Building an approximating system of Pfaffian equations

Let G be an open neighbourhood of the unit cube I^n and $f : G \longrightarrow \mathbb{R}$ a Pfaffian function of order r and degree (α, β).

As part of the description in Section §4.2 of a certain cylindrical cell decomposition $\mathcal{D}$ of I^n compatible with $Zer(f)$, we have defined sets of the kind $\Omega^{(i)}$, $0 \le i \le n-1$, and noted in particular, that a proof of an upper bound for the number of points in $\Omega^{(n-1)}$ would also imply a similar upper bound for the number of cells in $\mathcal{D}$.

In order to estimate the cardinality of $\Omega^{(n-1)}$ we will construct a system of equations with Pfaffian functions defined over some nonstandard extension of $\mathbb{R}$, in many variables including $X_1, \dots, X_n$, such that:

- the system has a finite number of roots;
- each point from $\Omega^{(n-1)}$ is "approximated" by the projection on $X_1, \dots, X_n$ of an appropriate root of the system, in particular, the cardinality of $\Omega^{(n-1)}$ does not exceed the number of roots.

Then we will estimate the number of roots using the Khovanskii's bound.

Define integers $s_0 = 0$ and recursively $s_{i+1} = 3s_i + i + 1$ for $0 \le i \le n-2$. Introduce new variables $Y_1, Y_2, \dots, Y_{s_{n-1}}$ and consider positive infinitesimals

$$\varepsilon_0 \gg \varepsilon_1 \gg \cdots \gg \varepsilon_{n-1} \gg \delta_1 \gg \delta_2 \gg \cdots \gg \delta_{n-1} \gg$$

$$\nu_0 \gg \nu_1 \gg \nu_1^{[1]} \gg \mu_1 \gg \nu_2 \gg \nu_2^{[1]} \gg \cdots \gg \nu_2^{[3]} \gg \mu_2 \gg \nu_3 \gg \cdots$$

$$\cdots \gg \mu_{n-2} \gg \nu_{n-1} \gg \nu_{n-1}^{[1]} \gg \cdots \gg \nu_{n-1}^{[s_{n-2}+n-1]} \gg \mu_{n-1} > 0.$$

Recall that $T^{(m)} = (T_1, \dots, T_m)$; in short, we write $T^{(m)} = 0$ for the expression

$$T_1 = \cdots = T_m = 0.$$

We also need to define the following polynomials:

$$p_1(X) := X_1(X_1 - 1)(X_1 + 1 - \varepsilon_0)(X_1 - 1 + \varepsilon_0),$$

and for each k, $2 \le k \le n$,

$$p_k(Y^{(s_{k-2})}, X) := \prod_{j=1}^{k} (X_j - \mu_{k-1})(X_j - 1 + \mu_{k-1}) \cdot \prod_{m=1}^{s_{k-2}} (Y_m - \mu_{k-1})(Y_m - 1 + \mu_{k-1}).$$

We introduce Pfaffian functions $g^{(i)}$ by induction on i according to the following formulas. At step i, variables $X_{i+2}, \dots, X_n$ are to be considered as parameters of these expressions. Explanatory comments are also provided for some stages of the initial steps of this inductive construction; analogous explanations are valid for the generalizations of these functions appearing in later steps.

Step i = 0

$f^{(0)}(X) := \big(X_1(X_1 - 1)f(X)\big)^2$

$h_0^{(0)}(X) := (f^{(0)}(X) - \nu_0)^2$

If $f(X) \not\equiv 0$ then $h_0^{(0)}(X) = 0$ defines a finite set of points which include $O(\nu_0)$-approximations to solutions of $X_1(X_1 - 1)f(X) = 0$.

$h_1^{(0)}(X) := (f^{(0)}(X))^2 + (p_1(X))^2$

If $f(X) \equiv 0$ then $h_1^{(0)}(X) = 0$ defines the set $\Omega^{(0)} = \{1, \varepsilon_0, 1 - \varepsilon_0, 1\}$.

$g^{(0)}(X) := h_0^{(0)}(X) \cdot h_1^{(0)}(X)$

The equation $g^{(0)} = 0$ defines infinitesimal approximations to the set $\Omega^{(0)}$ and to the (parametric) curve $\Gamma^{(1)}$.

Step i = 1

$\Delta^{(0)} := \left(\frac{\partial g^{(0)}}{\partial X_1}\right)^2 + \left(\frac{\partial g^{(0)}}{\partial X_2}\right)^2$

$h_0^{(1)}(X) := (g^{(0)}(X) - \nu_1)^2 + \left(\left(\frac{\partial g^{(0)}}{\partial X_1}\right)^2 - \nu_1^{[1]}\Delta^{(0)}\right)^2$

Since $\nu_1 > 0$ is infinitesimal (rel. to $\mathbb{R}$), the set $\{g^{(0)} = \nu_1\}$ is a smooth hypersurface, on which the gradient vector of $g^{(0)}$ does not vanish. Moreover, each point of $\{g^{(0)} = 0\}$ (and hence each point of $\Gamma^{(1)}$) is infinitely (rel. to $\mathbb{R}$) close to some point on this hypersurface. The equation $h_0^{(1)} = 0$ defines a finite set of points on this hypersurface for which their normalized gradient vectors are parallel to the unit vector $(\sqrt{\nu_1^{[1]}}, \sqrt{1-\nu_1^{[1]}})$. Each of these points is $O(\nu_1^{[1]})$-close to a corresponding element from the set of critical points of the projection of $\{g^{(0)} = \nu_1\}$ on $\{X_1 = 0\}$ (obtained by replacing in the equation $h_0^{(1)}$ above, the infinitesimal $\nu_1^{[1]}$ by 0), which in turn is infinitely (rel. to $\mathbb{R}$) close to a corresponding point of local extrema of the coordinate function X_2 on $\{g^{(0)} = 0\}$ (and hence also on $\Gamma^{(1)} \cap \hat{I}^2$).

$h_1^{(1)}(X) := g^{(0)}(X_1, \ldots, X_n) + g^{(0)}(X_1 - \delta_{n-1}, X_2, \ldots, X_n)$

Intersecting the curve $\{g^{(0)} = 0\}$ with its infinitesimal shift along X_1-coordinate produces a finite (parameterized) subset of $\{g^{(0)} = 0\}$ which approximates the set of ramification points of $\Gamma^{(1)}$.

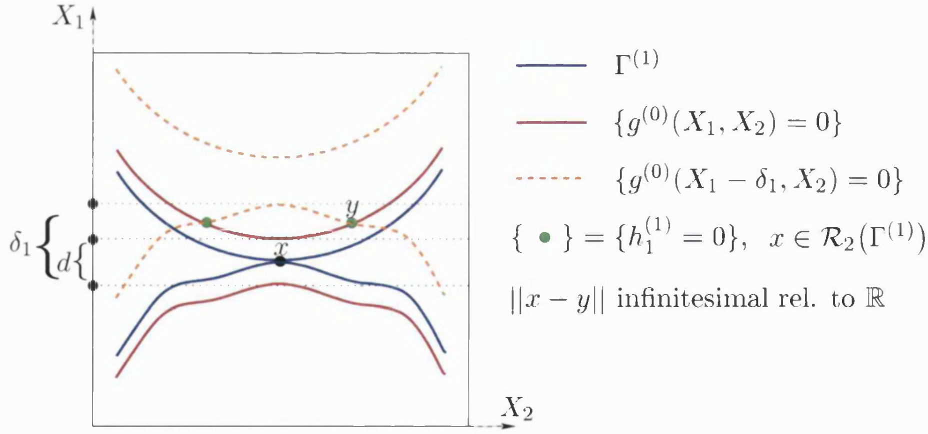


Figure 6-1: Approximations to ramification points of $\Gamma^{(1)}$ are defined as points of intersection of the curve $\{g^{(0)} = 0\}$ with its infinitesimal δ_1-shift along X_1-coordinate. Notice that, for this method to work, we need $\delta_1 \gg d = O(\nu_0)$.

$h_2^{(1)}(X) := (g^{(0)}(X))^2 + (p_2(X))^2$

The equation $h_2^{(1)} = 0$ defines approximations to special points (rel. to X_2-coordinate) of $\Gamma^{(1)}$ which lie on the boundary $\tilde{I}^2 = I^2 \setminus \hat{I}^2$ of the unit cube I^2.

$h^{(1)}(X) := h_0^{(1)} \cdot h_1^{(1)} \cdot h_2^{(1)}$
The equation $h^{(1)} = 0$ defines infinitesimal approximations to the set of special points $\Omega_s^{(1)}$ of the curve $\Gamma^{(1)}$ relative to X_2-coordinate.

$g_0^{(1)}(Y_1, X) := h_0^{(1)}(X) \cdot g^{(0)}(X) \cdot X_1^2 + h^{(1)}(Y_1, X_2, \dots, X_n)$
The projection along the variable Y_1 of the zeroset of $g_0^{(1)}(Y_1, X)$ includes infinitesimal approximations to points from $\Omega_0^{(1)}$. Note that in the function $h^{(1)}(Y_1, X_2, \dots, X_n)$ variable Y_1 stands for X_1. For any fixed values of parameters $X_3, \dots, X_n$ the zeroset of $h^{(1)}$ is finite and therefore the set $\{g^{(0)} = 0\} \cap \{h^{(1)} = 0\}$ reduces to an intersection of two finite unions of affine subspaces of complementary dimensions in 3-dimensional space and hence is finite. The set $\{h_0^{(1)}(X) = 0\}$ (which by definition is a subset of $\{h^{(1)}(X) = 0\}$) includes in particular points on $\{g^{(0)} = \nu_1\}$ that provide approximations to points of local extremum on $\Gamma^{(1)}$. But it is possible (since we are dealing with approximations) that a point of this kind, would not be included in the projection onto the X coordinates of the set $\{g^{(0)} = 0\} \cap \{h^{(1)} = 0\}$ (see Figure 6-2). To ensure that this does not happen we need to add points from $\{h_0^{(1)}(X) = 0\} \cap \{h^{(1)}(Y_1, X_2, \dots, X_n) = 0\}$.

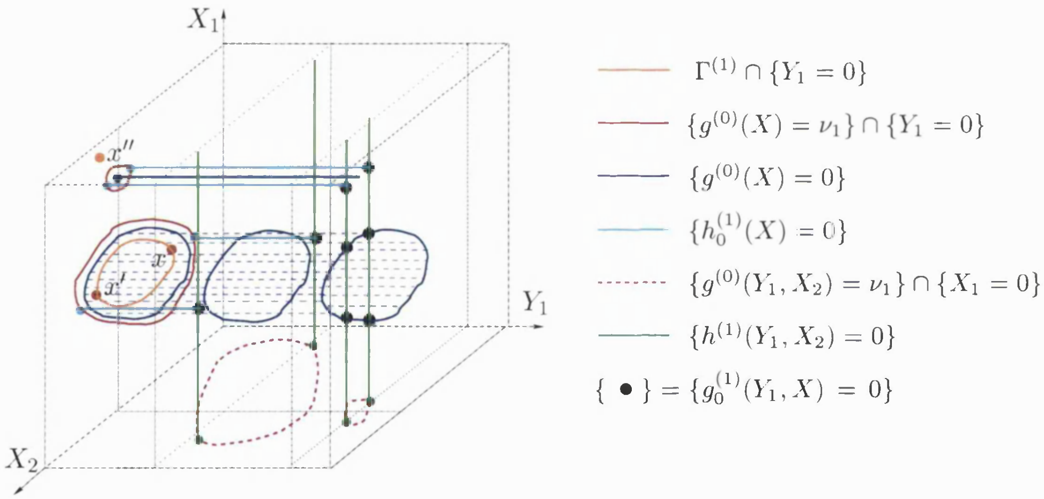


Figure 6-2: Projections onto $\{Y_1 = 0\}$ of solutions of the equation $g^{(0)}(X) \cdot X_1^2 + h^{(1)}(Y_1, X_2, \dots, X_n) = 0$ do not provide approximations to points x, x', x'' of local extremum of the curve $\Gamma^{(1)}$ (for the sake of clarity, $\Gamma^{(1)} \cap I_1^2$ has not been drawn).

$g_1^{(1)}(Y_1, X) := h_0^{(1)}(X) \cdot g^{(0)}(X) \cdot X_1^2 + h^{(1)}(Y_1, X_2 \pm \varepsilon_1, X_3, \dots, X_n)$
The projection along the variable Y_1 of the zeroset of $g_1^{(1)}(Y_1, X)$ includes infinitesimal approximations to points from $\Omega_1^{(1)}$.

$g^{(1)}(Y_1, X) := g_0^{(1)} \cdot g_1^{(1)}$
The equation $g^{(1)} = 0$ defines a finite set whose projection along the Y_1 variable includes infinitesimal approximations to $\Omega^{(1)}$ and to the (parametric) curve $\Gamma^{(2)}$.

$q_0^{(1)}(Y_1, X) := g_0^{(1)}(Y_1, X)$

$q_1^{(1)}(Y_1, X) := g_0^{(1)}(Y_1, X) \cdot (X_2^2 + h^{(1)}(X_1, Y_1, X_3, \dots, X_n))$

The equation $X_2^2 + h^{(1)}(X_1, Y_1, X_3, \dots, X_n) = 0$ defines a finite set of points in the space of coordinates Y_1, X_1, X_2, whose projection along the variable Y_1 includes infinitesimal approximations to all values appearing as X_1-coordinates of points from $\Omega_e^{(1)}$. The union of these points with solutions of the equation $g_0^{(1)} = 0$ make up the set $\{q_1^{(1)} = 0\}$. For variable X_3, the projection of this set along Y_1 includes projections of the curve $\Gamma_e^{(2)} \subset \Gamma_0^{(2)}$ onto the 2-plane X_1, X_3. The reason why we need to define this function will be explained shortly. We note that on the expense of adding just one new variable Y_2 we could have defined by the equation $g_0^{(1)}(Y_1, X_1, Y_2, \dots, X_n) + X_2^2 = 0$ a set whose projection along the variables Y_1, Y_2 includes projections of the curve $\Gamma_0^{(2)}$ (instead of just $\Gamma_e^{(2)}$) onto the 2-plane with coordinates X_1, X_3.

Step i = 2

$\Delta^{(1)} := \left(\frac{\partial g^{(1)}}{\partial Y_1}\right)^2 + \left(\frac{\partial g^{(1)}}{\partial X_1}\right)^2 + \left(\frac{\partial g^{(1)}}{\partial X_2}\right)^2 + \left(\frac{\partial g^{(1)}}{\partial X_3}\right)^2$

$$\hat{h}_0^{(2)}(Y_1, X) := (g^{(1)}(Y_1, X) - \nu_2)^2 + \left(\left(\frac{\partial g^{(1)}}{\partial Y_1}\right)^2 - \nu_2^{[1]}\Delta^{(1)}\right)^2 + \\ + \left(\left(\frac{\partial g^{(1)}}{\partial X_1}\right)^2 - \nu_2^{[2]}\Delta^{(1)}\right)^2 + \left(\left(\frac{\partial g^{(1)}}{\partial X_2}\right)^2 - \nu_2^{[3]}\Delta^{(1)}\right)^2$$

The equation $\hat{h}_0^{(1)} = 0$ defines a finite set of points in the coordinate space Y_1, X_1, X_2, X_3 that lie on the smooth hypersurface $\{g^{(1)} = \nu_2\}$, for which their normalized gradient vectors are parallel to the unit vector

$$\left(\sqrt{\nu_2^{[1]}}, \sqrt{\nu_2^{[2]}}, \sqrt{\nu_2^{[3]}}, \sqrt{1 - \nu_2^{[1]} - \nu_2^{[2]} - \nu_2^{[3]}}\right).$$

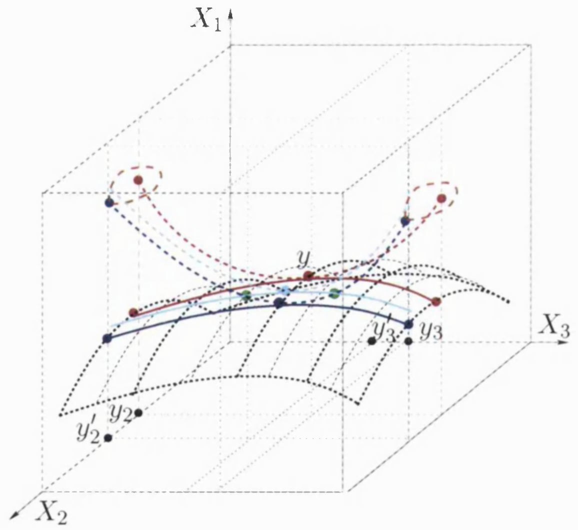
Projections of these points onto the X_3-axis approximate the set of critical values of the projection of $\{g^{(1)} = \nu_2\}$ on the same axis (unlike the previous step, the set of critical points of this projection map need not be finite). Projections along the variable Y_1 of points satisfying the equation $\hat{h}_0^{(2)} = 0$ provide infinitesimal approximations to points of local extrema of the coordinate function X_2 on $\Gamma^{(2)} \cap \hat{I}^3$.

$h_0^{(2)}(Y^{(2)}, X) := \hat{h}_0^{(2)}(Y_1, X) + Y_2^2$

$h_1^{(2)}(Y^{(2)}, X) := q_0^{(1)}(Y_1, X) + q_0^{(1)}(Y_2, X_1, X_2 - \delta_{n-2}, X_3, \dots, X_n)$

Intersecting the projection onto X coordinates of the curve $\{q_0^{(1)} = 0\}$ with the projection of its infinitesimal shift along X_2-coordinate produces a finite (parameterized) set in the space $\{Y_1 = Y_2 = 0\}$ whose projection onto $\{Y_1 = Y_2 = X_1 = X_2 = 0\}$ includes approximations for some, but unfortunately not all, of the values of X_3-coordinates of ramification points of $\Gamma^{(2)}$ (Figure 6-3 shows an example in which the curve $\Gamma^{(2)} \subset I^3 \cap \{X_2 = y_2\}$ has a ramification point and $\{h_1^{(2)} = 0\} = \emptyset$).

$h_2^{(2)}(Y^{(2)}, X) := q_1^{(1)}(Y_1, X) + q_1^{(1)}(Y_2, X_1 - \delta_{n-2}, X_2, \ldots, X_n)$
Intersecting the projection onto X coordinates of the curve $\{q_1^{(1)} = 0\}$ with the projection of its infinitesimal shift along X_1-coordinate produces a finite (parameterized) set in the space $\{Y_1 = Y_2 = 0\}$ whose projection onto $\{Y_1 = Y_2 = X_1 = X_2 = 0\}$ includes approximations for the remaining values of X_3-coordinates of ramification points of $\Gamma^{(2)}$. The examples in Figures 6-3 and 6-4 show the necessity of the introduction of the function $h_2^{(2)}$ in order to define values infinitely close to the set of all X_3-coordinates of ramification points of $\Gamma^{(2)}$. Because of the non-exact nature of our construction, is possible that some local geometric feature of the curve $\Gamma_e^{(2)}$, such as its possession of a ramification point, would not be passed on to the component of the projection along Y_1, Y_2 of the curve $\{g^{(1)} = 0\}$ that lies infinitely (rel. to $\mathbb{R}$) close to $\Gamma_e^{(2)}$ (see Figure 6-4); in this case the role of the curve $\{q_1^{(1)} = 0\}$ becomes crucial. Note that this sort of problematic behaviour is associated only with subsets of the curve $\Gamma_e^{(2)}$ (cf Figure 6-3). Finally, we note that the example shown in Figure 6-4 provides also some justification for the reasoning behind the specific choice of ordering among the infinitesimal elements present in this definition.



$\Gamma^{(1)}$ ------ Γ' —— Γ'' ....

C —— D ...... $\hat{C}$ —— $\hat{D}$ ......

$\Gamma' \subset \Gamma_e^{(2)}, \quad \Gamma'' \not\subset \Gamma_e^{(2)}$

$C \cup D \subset \pi(\{g^{(1)} = 0\})$

$\hat{C}$ = shifting C by δ_1 along X_1

$\hat{D}$ = shifting D by δ_1 along X_1

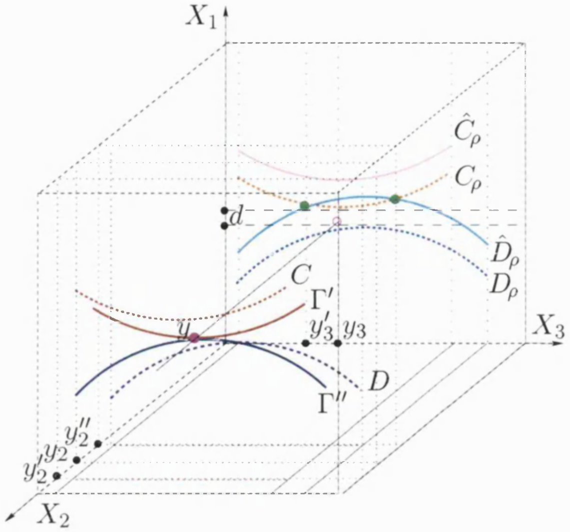
$\pi(\{h_2^{(2)} = 0\}) = \{\bullet\}$

$(y_2' - y_2) \ll \delta_1$

$(y_2' - y_2), (y_3 - y_3')$ infinit. rel. $\mathbb{R}$

Figure 6-3: The curve $C \cup D$ approximates $\Gamma' \cup \Gamma''$ and the value y_3', which appears as the X_3-coordinate of a point from $\{h_2^{(2)} = 0\}$, approximates the X_3-coordinate of the ramification point y of $\Gamma^{(2)}$. Here π denotes the projection map on the X-coordinates.

$h_3^{(2)}(Y^{(2)}, X) := (g^{(1)}(Y_1, X))^2 + (p_3(Y_1, X))^2 + Y_2^2$
The projection along the variables Y_1, Y_2 of the zeroset of $h_3^{(2)}$ contains approximations to special points (relative to X_3-coordinate) of $\Gamma^{(2)}$ which lie on the boundary $\tilde{I}^3$ of I^3.



$\Gamma' \cup \Gamma'' \subset \Gamma_e^{(2)} \cap \{X_2 = y_2\}$

$C \cup D \subset \pi(\{g^{(1)} = 0\})$

$C_\rho = \rho(C), \quad D_\rho = \rho(D)$

$C_\rho \cup D_\rho \subset \pi(\{q_1^{(1)} = 0\ )$

$(y_2' - y_2'') \ll \delta_1, \qquad d \ll \delta_1$

$\hat{C}_\rho =$ shifting C_ρ by δ_1 along X_1

$\hat{D}_\rho =$ shifting D_ρ by δ_1 along X_1

$\pi(\{h_2^{(2)} = 0\}) = \{\bullet\}$

Figure 6-4: The curve $C \cup D$ approximates $\Gamma' \cup \Gamma''$ and the value y_3', which appears as the X_3-coordinate of a point from $\{h_2^{(2)} = 0\}$, approximates the X_3-coordinate of the ramification point y of $\Gamma_e^{(2)}$. Here π denotes the projection map on the X-coordinates and ρ the projection map onto the plane with coordinates X_1, X_3.

$h^{(2)}(Y^{(2)}, X) := h_0^{(2)} \cdot h_1^{(2)} \cdot h_2^{(2)} \cdot h_3^{(2)}$
The equation $h^{(2)} = 0$ defines a finite set of points whose projection onto the X_3-axis contains infinitesimal approximations to all the special values relative to X_3-coordinate of the curve $\Gamma^{(2)}$.

$g_0^{(2)}(Y^{(5)}, X) := \hat{h}_0^{(2)}(Y_3, X) \cdot g^{(1)}(Y_3, X) \cdot (Y_3^2 + X_1^2 + X_2^2) + h^{(2)}(Y^{(2)}, Y_4, Y_5, X_3, \ldots, X_n)$
The projection along the variables $Y_1, \ldots, Y_5$ of the zeroset of $g_0^{(2)}(Y^{(5)}, X)$ includes infinitesimal approximations to points from $\Omega_0^{(2)}$.

$$g_1^{(2)}(Y^{(5)}, X) := \hat{h}_0^{(2)}(Y_3, X) \cdot g^{(1)}(Y_3, X) \cdot (Y_3^2 + X_1^2 + X_2^2) + \\ + h^{(2)}(Y^{(2)}, Y_4, Y_5, X_3 \pm \varepsilon_2, X_4, \ldots, X_n)$$
The projection along the variables $Y_1, \ldots, Y_5$ of the zeroset of $g_1^{(2)}(Y^{(5)}, X)$ includes infinitesimal approximations to points from $\Omega_1^{(2)}$.

$g^{(2)}(Y^{(5)}, X) := g_0^{(2)} \cdot g_1^{(2)}$
The equation $g^{(2)} = 0$ defines a finite set whose projection along the variables $Y_1, \ldots, Y_5$ includes infinitesimal approximations to $\Omega^{(2)}$ and to the (parametric) curve $\Gamma^{(3)}$.

$q_0^{(2)}(Y^{(5)}, X) := g_0^{(2)}(Y^{(5)}, X)$

$q_1^{(2)}(Y^{(5)}, X) := g_0^{(2)}(Y^{(5)}, X) \cdot ((Y_3)^2 + X_1^2 + X_3^2) + h^{(2)}(Y^{(2)}, Y_4, X_2, Y_5, X_4, \ldots, X_n))$

$q_2^{(2)}(Y^{(5)}, X) := g_0^{(2)}(Y^{(5)}, X) \cdot ((Y_3)^2 + X_2^2 + X_3^2) + h^{(2)}(Y^{(2)}, X_1, Y_4, Y_5, X_4, \ldots, X_n))$

For $j = 0, 1, 2$, the equation $q_j^{(2)} = 0$ defines a finite set of points in the space of coordinates $Y^{(5)}, X_1, X_2, X_3$ which includes the zeroset of $g_0^{(2)}$ and whose projection along the variables $Y_1, \dots, Y_5$ contains infinitesimal approximations to all values appearing as X_{3-j}-coordinates of points from $\Omega_e^{(2)}$. Thus, for variable X_4, the projection of this set along $Y_1, \dots, Y_5$ contains projections of the curve $\Gamma_e^{(3)}$ onto the 2-plane equipped with coordinates X_{3-j}, X_4.

Suppose that on the **step i**, for $\mathbf{0 \leq i \leq n-3}$, we have defined the functions

$g^{(i)}(Y^{(s_i)}, X)$ and $q_j^{(i)}(Y^{(s_i)}, X)$, $0 \leq j \leq i$.

Step i + 1

$$\Delta^{(i)} = \sum_{1 \leq j \leq s_i} \left(\frac{\partial g^{(i)}}{\partial Y_j}\right)^2 + \sum_{1 \leq j \leq i+2} \left(\frac{\partial g^{(i)}}{\partial X_j}\right)^2$$

$$\begin{aligned}
\hat{h}_0^{(i+1)}(Y^{(s_i)}, X) := {} & (g^{(i)}(Y^{(s_i)}, X) - \nu_{i+1})^2 + \\
& + \left(\left(\frac{\partial g^{(i)}}{\partial Y_1}\right)^2 - \nu_{i+1}^{[1]} \Delta^{(i)}\right)^2 + \cdots + \left(\left(\frac{\partial g^{(i)}}{\partial Y_{s_i}}\right)^2 - \nu_{i+1}^{[s_i]} \Delta^{(i)}\right)^2 + \\
& + \left(\left(\frac{\partial g^{(i)}}{\partial X_1}\right)^2 - \nu_{i+1}^{[s_i+1]} \Delta^{(i)}\right)^2 + \cdots + \left(\left(\frac{\partial g^{(i)}}{\partial X_{i+1}}\right)^2 - \nu_{i+1}^{[s_i+i+1]} \Delta^{(i)}\right)^2
\end{aligned}$$

$$h_0^{(i+1)}(Y^{(2s_i)}, X) := \hat{h}_0^{(i+1)}(Y^{(s_i)}, X) + \sum_{j=s_i+1}^{2s_i} Y_j^2$$

$$\begin{aligned}
h_1^{(i+1)}(Y^{(2s_i)}, X) := {} & q_0^{(i)}(Y^{(s_i)}, X_1, \dots, X_n) + \\
& + q_0^{(i)}(Y_{s_i+1}, \dots, Y_{2s_i}, X_1, \dots, X_i, X_{i+1} - \delta_{n-i-1}, X_{i+2}, \dots, X_n)
\end{aligned}$$

$$\begin{aligned}
h_2^{(i+1)}(Y^{(2s_i)}, X) := {} & q_1^{(i)}(Y^{(s_i)}, X_1, \dots, X_n) + \\
& + q_1^{(i)}(Y_{s_i+1}, \dots, Y_{2s_i}, X_1, \dots, X_{i-1}, X_i - \delta_{n-i-1}, X_{i+1}, \dots, X_n)
\end{aligned}$$

$$\begin{aligned}
h_3^{(i+1)}(Y^{(2s_i)}, X) := {} & q_2^{(i)}(Y^{(s_i)}, X_1, \dots, X_n) + \\
& + q_2^{(i)}(Y_{s_i+1}, \dots, Y_{2s_i}, X_1, \dots, X_{i-2}, X_{i-1} - \delta_{n-i-1}, X_i, X_{i+1}, \dots X_n)
\end{aligned}$$

$\dots \quad \dots \quad \dots$

$$\begin{aligned}
h_j^{(i+1)}(Y^{(2s_i)}, X) := {} & q_{j-1}^{(i)}(Y^{(s_i)}, X_1, \dots, X_n) + \\
& + q_{j-1}^{(i)}(Y_{s_i+1}, \dots, Y_{2s_i}, X_1, \dots, X_{i+1-j}, X_{i+2-j} - \delta_{n-i-1}, X_{i+3-j}, \dots, X_n)
\end{aligned}$$

$\dots \quad \dots \quad \dots$

$$h_{i+1}^{(i+1)}(Y^{(2s_i)}, X) := q_i^{(i)}(Y^{(s_i)}, X) + q_i^{(i)}(Y_{s_i+1}, \dots, Y_{2s_i}, X_1 - \delta_{n-i-1}, X_2, \dots, X_n)$$

$$h_{i+2}^{(i+1)}(Y^{(2s_i)}, X) := (g^{(i)}(Y^{(s_i)}, X))^2 + (p_{i+2}(Y^{(s_i)}, X))^2 + \sum_{j=s_i+1}^{2s_i} Y_j^2$$

$$h^{(i+1)}(Y^{(2s_i)}, X) := h_0^{(i+1)} \cdots h_{i+1}^{(i+1)} h_{i+2}^{(i+1)}$$

$$\begin{aligned}
g_0^{(i+1)}(Y^{(s_{i+1})}, X) := {} & (Y_{2s_i+1}^2 + \cdots + Y_{3s_i}^2 + X_1^2 + \cdots + X_{i+1}^2) \cdot \\
& \cdot \hat{h}_0^{(i+1)}(Y_{2s_i+1}, \dots, Y_{3s_i}, X) \cdot g^{(i)}(Y_{2s_i+1}, \dots, Y_{3s_i}, X) + \\
& + h^{(i+1)}(Y^{(2s_i)}, Y_{3s_i+1}, \dots, Y_{3s_i+i+1}, X_{i+2}, \dots, X_n)
\end{aligned}$$

$$
\begin{aligned}
g_1^{(i+1)}(Y^{(s_{i+1})},X) := {} & (Y_{2s_i+1}^2+\cdots+Y_{3s_i}^2+X_1^2+\cdots+X_{i+1}^2)\cdot \\
& \cdot \hat{h}_0^{(i+1)}(Y_{2s_i+1},\ldots,Y_{3s_i},X)\cdot g^{(i)}(Y_{2s_i+1},\ldots,Y_{3s_i},X)+ \\
& +h^{(i+1)}(Y^{(2s_i)},Y_{3s_i+1},\ldots,Y_{3s_i+i+1},X_{i+2}\pm\varepsilon_{i+1},\ldots,X_n)
\end{aligned}
$$

$$g^{(i+1)}(Y^{(s_{i+1})},X) := g_0^{(i+1)}\cdot g_1^{(i+1)}$$

$$q_0^{(i+1)}(Y^{(s_{i+1})},X) := g_0^{(i+1)}(Y^{(s_{i+1})},X)$$

$$
\begin{aligned}
q_1^{(i+1)}(Y^{(s_{i+1})},X) := {} & \big(g_0^{(i+1)}(Y^{(s_{i+1})},X)\big)\cdot \\
& \cdot\big(((Y_{2s_i+1})^2+\cdots+(Y_{3s_i})^2+X_1^2+\cdots+X_i^2+X_{i+2}^2+\cdots+X_{i+2}^2)+ \\
& +h^{(i+1)}(Y^{(2s_i)},Y_{3s_i+1},\ldots,Y_{3s_i+i},X_{i+1},Y_{3s_i+i+1},\ldots,Y_{3s_i+i+1},X_{i+3},\ldots,X_n)\big)
\end{aligned}
$$

$$
\begin{aligned}
q_2^{(i+1)}(Y^{(s_{i+1})},X) := {} & \big(g_0^{(i+1)}(Y^{(s_{i+1})},X)\big)\cdot \\
& \cdot\big(((Y_{2s_i+1})^2+\cdots+(Y_{3s_i})^2+X_1^2+\cdots+X_{i-1}^2+X_{i+1}^2+\cdots+X_{i+2}^2)+ \\
& +h^{(i+1)}(Y^{(2s_i)},Y_{3s_i+1},\ldots,Y_{3s_i+i-1},X_i,Y_{3s_i+i},\ldots,Y_{3s_i+i+1},X_{i+3},\ldots,X_n)\big)
\end{aligned}
$$

$\ldots \quad \ldots \quad \ldots$

$$
\begin{aligned}
q_j^{(i+1)}(Y^{(s_{i+1})},X) := {} & \big(g_0^{(i+1)}(Y^{(s_{i+1})},X)\big)\cdot \\
& \cdot\big(((Y_{2s_i+1})^2+\cdots+(Y_{3s_i})^2+X_1^2+\cdots+X_{i+1-j}^2+X_{i+3-j}^2+\cdots+X_{i+2}^2)+ \\
& +h^{(i+1)}(Y^{(2s_i)},Y_{3s_i+1},\ldots,Y_{3s_i+i+1-j},X_{i+2-j},Y_{3s_i+i+2-j},\ldots,Y_{3s_i+i+1},X_{i+3},\ldots,X_n)\big)
\end{aligned}
$$

$\ldots \quad \ldots \quad \ldots$

$$
\begin{aligned}
q_{i+1}^{(i+1)}(Y^{(s_{i+1})},X) := {} & \big(g_0^{(i+1)}(Y^{(s_{i+1})},X)\big)\cdot\big(((Y_{2s_i+1})^2+\cdots+(Y_{3s_i})^2+X_2^2+\cdots+X_{i+2}^2)+ \\
& +h^{(i+1)}(Y^{(2s_i)},Y_{3s_i+1},\ldots,Y_{3s_i},X_1,Y_{3s_i+1},\ldots,Y_{3s_i+i+1},X_{i+3},\ldots,X_n)\big)
\end{aligned}
$$

Step $\mathbf{i=n-1}$

$$\Delta^{(n-2)} = \sum_{j=1}^{s_{n-2}}\left(\frac{\partial g^{(n-2)}}{\partial Y_j}\right)^2+\sum_{1\le j\le n}\left(\frac{\partial g^{(n-2)}}{\partial X_j}\right)^2$$

$$
\begin{aligned}
\hat{h}_0^{(n-1)}(Y^{(s_{n-2})},X) := {} & \big(g^{(n-2)}(Y^{(s_{n-2})},X)-\nu_{n-1}\big)^2+ \\
& +\left(\left(\frac{\partial g^{(n-2)}}{\partial Y_1}\right)^2-\nu_{n-1}^{[1]}\Delta^{(n-2)}\right)^2+\cdots+\left(\left(\frac{\partial g^{(n-2)}}{\partial Y_{s_{n-2}}}\right)^2-\nu_{n-1}^{[s_{n-2}]}\Delta^{(n-2)}\right)^2+ \\
& +\left(\left(\frac{\partial g^{(n-2)}}{\partial X_1}\right)^2-\nu_{n-1}^{[s_{n-2}+1]}\Delta^{(n-2)}\right)^2+\cdots+\left(\left(\frac{\partial g^{(n-2)}}{\partial X_{n-1}}\right)^2-\nu_{n-1}^{[s_{n-2}+n-1]}\Delta^{(n-2)}\right)^2
\end{aligned}
$$

$$h_0^{(n-1)}(Y^{(2s_{n-1})},X) := \hat{h}_0^{(n-1)}(Y^{(s_{n-2})},X)+\sum_{j=s_{n-2}+1}^{2s_{n-2}}Y_j^2$$

For j, $1\le j\le n-1$ define

$$
\begin{aligned}
h_j^{(n-1)}(Y^{(2s_{n-2})},X) := {} & q_{j-1}^{(n-2)}(Y^{(s_{n-2})},X_1,\ldots,X_n)+ \\
& +q_{j-1}^{(n-2)}(Y_{s_{n-2}+1},\ldots,Y_{2s_{n-2}},X_1,\ldots,X_{n-1-j},X_{n-j}-\delta_1,X_{n-j+1},\ldots,X_n)
\end{aligned}
$$

$$h_n^{(n-1)}(Y^{(2s_{n-2})},X) := (g^{(n-2)}(Y^{(s_{n-2})},X))^2+(p_n(Y^{(s_{n-2})},X))^2+\sum_{j=s_{n-2}+1}^{2s_{n-2}}Y_j^2$$

$$
\begin{aligned}
h^{(n-1)}(Y^{(2s_{n-2})},X) &:= h_0^{(n-1)} \cdots h_{n-1}^{(n-1)} h_n^{(n-1)} \\
g_0^{(n-1)}(Y^{(s_{n-1})},X) &:= (Y^2_{2s_{n-2}+1} + \cdots + Y^2_{3s_{n-2}} + X_1^2 + \cdots + X^2_{n-1}) \cdot \\
&\quad \cdot \hat{h}_0^{(n-1)}(Y_{2s_{n-2}+1}, \ldots, Y_{3s_{n-2}}, X) \cdot g^{(n-2)}(Y_{2s_{n-2}+1}, \ldots, Y_{3s_{n-2}}, X) + \\
&\quad + h^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \ldots, Y_{3s_{n-2}+n-1}, X_n) \\
g_1^{(n-1)}(Y^{(s_{n-1})},X) &:= (Y^2_{2s_{n-2}+1} + \cdots + Y^2_{3s_{n-2}} + X_1^2 + \cdots + X^2_{n-1}) \cdot \\
&\quad \cdot \hat{h}_0^{(n-1)}(Y_{2s_{n-2}+1}, \ldots, Y_{3s_{n-2}}, X) \cdot g^{(n-2)}(Y_{2s_{n-2}+1}, \ldots, Y_{3s_{n-2}}, X) + \\
&\quad + h^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \ldots, Y_{3s_{n-2}+n-1}, X_n \pm \varepsilon_{n-1}) \\
g^{(n-1)}(Y^{(s_{n-1})},X) &:= g_0^{(n-1)} \cdot g_1^{(n-1)}
\end{aligned}
$$

This concludes the inductive definition of the Pfaffian function $g^{(n-1)}$.

6.2.2 Defining points infinitely close to points from $\Omega^{(i)}$

We want to show that the equation $g^{(n-1)} = 0$ defines a finite set of points whose projection onto the coordinate subspace $X_1, \ldots, X_n$ provides infinitesimal approximations to points from $\Omega^{(n-1)}$. We call the reader's attention to the fact that a number of arguments which we employ in the proof of this result, have allready appeared (possibly with some modifications) in the proof of the subanalyticity of $\Omega^{(n-1)}$ in Section §5.1.

Consider the sequence of ordered fields $\mathbb{R} = \mathbb{R}_0 \subset \mathbb{R}_1 \subset \cdots \subset \mathbb{R}_n = \mathbb{R}_\varepsilon$ in which the field $\mathbb{R}_{k+1}$, $0 \le k \le n-1$, contains a positive element ε_k infinitesimal relative to $\mathbb{R}_k$, and let $\mathbb{R}_{n+j}$, $1 \le j \le n$, be a non-standard extension of $\mathbb{R}_n$ containing the infinitesimals

$$\delta_{n-j+1}, \ldots, \delta_{n-1}, \mu_1, \ldots, \mu_{j-1}, \nu_0, \ldots, \nu_{j-1}^{[s_{j-2}+j-1]}.$$

Suppose that $x \in \mathbb{R}_l$, $0 \le l \le 2n$. Denote by $st_m(x)$, $0 \le m \le l$, the standard part of x relative to the field $\mathbb{R}_m$.

For each i, $0 \le i \le n-1$, let the space $\mathbb{R}^{s_i+i+1}_{n+i+1}$ be equipped with variables $Y^{(s_i)}, X^{(i+1)}$. Let

$$\pi_i : \quad \mathbb{R}^{s_i+i+1}_{n+i+1} \longrightarrow \mathbb{R}^{i+1}_{n+i+1}$$

denote the projection map onto the subspace $\{Y^{(s_i)} = 0\}$ and for $k = 1, \ldots, i+1$ define

$$\rho_k^{(i)} : \quad \mathbb{R}^{i+1}_{n+i+1} \longrightarrow \mathbb{R}_{n+i+1}$$

to be the projection map onto the space $\{X_1 = \cdots = X_{k-1} = X_{k+1} = \cdots = X_{i+1} = 0\}$.

Lemma 6.2.3. *The set $\{g^{(n-1)} = 0\} \subset \mathbb{R}^{n+s_{n-1}}_{2n}$ consists of a finite number of points.*

Proof. We shall proceed by induction on i, $0 \le i \le n-1$.

First consider the case $i = 0$. Notice that if $f^{(0)}(X)$ is identically zero, then the set $\{h_0^{(0)} = 0\} = \emptyset$ and thus, $\{g^{(0)} = 0\} = \{0, \varepsilon_0, 1, 1 - \varepsilon_0\}$. If $\{f^{(0)}(X) = 0\}$ is a finite collection of points then $\{h_0^{(0)} = 0\}$ is also a finite set of points, with the property that for every point in $\{f^{(0)}(X) = 0\}$, there is a point in $\{h_0^{(0)} = 0\}$ infinitely (rel. to $\mathbb{R}$) close to it. Moreover, $\{h_1^{(0)} = 0\} \subset \{f^{(0)}(X) = 0\}$ and hence, $\{g^{(0)} = 0\} \subset \mathbb{R}_1$ consists of a finite number of points.

Suppose that the sets determined by $g^{(i)}$ for $i < n-1$ are finite. We show that this is also the case for $i = n-1$.

We start by proving that $\{h^{(n-1)} = 0\}$ consists of a finite number of points. We first consider the set $\{h_0^{(n-1)} = 0\}$. According to the formula,

$$h_0^{(n-1)}(Y^{(s_{n-2})}, X) := \left(g^{(n-2)}(Y^{(s_{n-2})}, X) - \nu_{n-1}\right)^2 + \textstyle\sum_{j=s_{n-2}+1}^{2s_{n-2}} Y_j^2 +$$
$$+\left(\left(\tfrac{\partial g^{(n-2)}}{\partial Y_1}\right)^2 - \nu_{n-1}^{[1]}\Delta^{(n-2)}\right)^2 + \cdots + \left(\left(\tfrac{\partial g^{(n-2)}}{\partial Y_{s_{n-2}}}\right)^2 - \nu_{n-1}^{[s_{n-2}]}\Delta^{(n-2)}\right)^2 +$$
$$+\left(\left(\tfrac{\partial g^{(n-2)}}{\partial X_1}\right)^2 - \nu_{n-1}^{[s_{n-2}+1]}\Delta^{(n-2)}\right)^2 + \cdots + \left(\left(\tfrac{\partial g^{(n-2)}}{\partial X_{n-1}}\right)^2 - \nu_{n-1}^{[s_{n-2}+n-1]}\Delta^{(n-2)}\right)^2,$$

where

$$\Delta^{(n-2)} = \sum_{j=1}^{s_{n-2}} \left(\frac{\partial g^{(n-2)}}{\partial Y_j}\right)^2 + \sum_{1 \le j \le n} \left(\frac{\partial g^{(n-2)}}{\partial X_j}\right)^2.$$

We prove by induction on l that the variety

$$G^{(l)} = \Big\{g^{(n-2)} - \nu_{n-1} = Y_{s_{n-2}+1} = \cdots = Y_{2s_{n-2}} =$$
$$= \left(\frac{\partial g^{(n-2)}}{\partial Y_1}\right)^2 - \nu_{n-1}^{[1]}\Delta^{(n-2)} = \cdots = \left(\frac{\partial g^{(n-2)}}{\partial Z_l}\right)^2 - \nu_{n-1}^{[l]}\Delta^{(n-2)} = 0\Big\}$$

has the dimension not exceeding $s_{n-2}+n-l-1$, where $0 \le l \le s_{n-2}+n-1$ and Z_l runs through all variables $Y_1, \ldots, Y_{s_{n-2}}, X_1, \ldots, X_{n-1}$. Base of the induction, for $l = 0$, is true since $\{g^{(n-2)} - \nu_{n-1} = 0\}$ is a (smooth) hypersurface in the $(s_{n-2}+n)$-dimensional space $\{Y_{s_{n-2}+1} = \cdots = Y_{2s_{n-2}} = 0\}$, due to ν_{n-1} being transcentental relative to the field $\mathbb{R}_{2n-1}$ over which the function $g^{(n-2)}$ is defined (see Corollary A.7).

Suppose the inductive hypothesis is true for l, i.e. $\dim(G^{(l)}) \le s_{n-2}+n-l-1$, we prove it for $l+1$.

Observe that the function

$$L^{(l+1)} = \left(\frac{\partial g^{(n-2)}}{\partial Z_{l+1}}\right)^2 - \nu_{n-1}^{[l+1]}\Delta^{(n-2)}$$

does not vanish on any subset of the full dimension of $G^{(l)}$. Indeed, if it did, then $L^{(l+1)}$ would vanish on some component C of $G^{(l)}$, that is defined over a smaller field $\mathbb{R}_{\nu_{n-1},\ldots,\nu_{n-1}^{[l]}}$ which extends $\mathbb{R}_{2n-1}$ and includes the elements $\nu_{n-1}, \ldots, \nu_{n-1}^{[l]}$ but not

$\nu_{n-1}^{[l+1]}$. Hence, there would be a point $x \in G^{(l)}$ defined over the field not including $\nu_{n-1}^{[l+1]}$ at which $L^{(l+1)}$ would also vanish. Since x is a point on the smooth hypersurface $\{g^{(n-2)} - \nu_{n-1} = 0\}$, the value $\Delta^{(n-2)}(x) \neq 0$, thus

$$\left(\frac{\partial g^{(n-2)}}{\partial Z_{l+1}}\right)^2 (x)$$

is, on one hand, an element from the field not including $\nu_{n-1}^{[l+1]}$, but on the other, an infinitesimal relative to that field. This contradiction shows that $L^{(l+1)}$ does not vanish on any subset of the full dimension of $G^{(l)}$, therefore

$$\dim(G^{(l+1)}) \leq \dim(G^{(l)}) - 1.$$

Applying the result to $l = s_{n-2} + n - 1$ we get that the set $G^{(s_{n-2}+n-1)} = \{h_0^{(n-1)} = 0\}$ has the dimension not exceeding 0.

Secondly, we prove by induction on n that $\{h_j^{(n-1)} = 0\}$ for $1 \leq j \leq n-1$ consists of a finite number of points. According to the formula,

$$h_j^{(n-1)} := q_{j-1}^{(n-2)}(Y^{(s_{n-2})}, X_1, \ldots, X_n) +$$

$$q_{j-1}^{(n-2)}(Y_{s_{n-2}+1}, \ldots, Y_{2s_{n-2}}, X_1, \ldots, X_{n-j-1}, X_{n-j} - \delta_1, X_{n-j+1}, \ldots, X_n),$$

where

$$q_{j-1}^{(n-2)}(Y^{(s_{n-2})}, X) = g_0^{(n-2)}(Y^{(s_{n-2})}, X) \cdot \bar{q}_{j-1}^{(n-2)}(Y^{(s_{n-2})}, X),$$

$$\bar{q}_{j-1}^{(n-2)}(Y^{(s_{n-2})}, X) = (Y^2_{2s_{n-3}+1} + \cdots + Y^2_{3s_{n-3}} + X_1^2 + \cdots + X^2_{n-1-j} + X^2_{n+1-j} + \cdots + X^2_{n-1}) +$$

$$+ h^{(n-2)}(Y^{(2s_{n-3})}, Y_{3s_{n-3}+1}, \ldots, Y_{3s_{n-3}+n-1-j}, X_{n-j}, Y_{3s_{n-3}+n+1-j}, \ldots, Y_{3s_{n-3}+n-2}, X_n).$$

Due to the inductive hypothesis, for any fixed value of X_n-coordinate, the set defined by $\bar{q}_{j-1}^{(n-2)}$ reduces to an intersection of finite unions of affine subspaces of complementary dimension, and hence is finite. Thus, the sets

$$Q_1 = \{q_{j-1}^{(n-2)}(Y_1, \ldots, Y_{s_{n-2}}, X_1, \ldots, X_n) = 0\}$$

and

$$Q_2 = \{q_{j-1}^{(n-2)}(Y_{s_{n-2}+1}, \ldots, Y_{2s_{n-2}}, X_1, \ldots, X_{n-j} - \delta_1, \ldots, X_n) = 0\}$$

are semi-Pfaffian curves in the $(s_{n-2} + n)$-dimensional spaces equipped with coordinates $Y_1, \ldots, Y_{s_{n-2}}, X_1, \ldots, X_n$ and $Y_{s_{n-2}+1}, \ldots, Y_{2s_{n-2}}, X_1, \ldots, X_n$ respectively. Consider the projections $\hat{Q}_1$, $\hat{Q}_2$ of Q_1, Q_2 respectively on the subspace of coordinates

$X_1, \ldots, X_n$, which are sub-Pfaffian curves. Observe that $\hat{Q}_2$ is obtained from $\hat{Q}_1$ by a δ_1-shift along the coordinate X_{n-j}. We prove that the intersection $\hat{Q}_1 \cap \hat{Q}_2$ consists of a finite number of points. Indeed, suppose that $\dim(\hat{Q}_1 \cap \hat{Q}_2) = 1$. Then there is a point $x \in \hat{Q}_1 \cap \hat{Q}_2$ defined over the field $\mathbb{R}_{2n-1}$ not including δ_1. This contradicts to x being a δ_1-shift of another point defined over $\mathbb{R}_{2n-1}$ (a field not including δ_1), namely of

$$\hat{Q}_1 \cap \{X_1 = x_1, \ldots, X_{n-j-1} = x_{n-j-1}, X_{n-j+1} = x_{n-j+1}, \ldots, X_n = x_n\}.$$

It follows that the set $Q_1 \cap Q_2$, if non-empty, is a transversal intersection of two s_{n-2}-dimensional planes in $2s_{n-2}$-dimensional space of coordinates $Y_1, \ldots, Y_{2s_{n-2}}$, and, therefore, is zero-dimensional.

Finally we show that $\{h_n^{(n-1)} = 0\}$ is zero-dimensional. According to the formula,

$$h_n^{(n-1)} := (g^{(n-2)}(Y^{(s_{n-2})}, X))^2 + \left(p_n(Y^{(s_{n-2})}, X)\right)^2 + \sum_{s_{n-2}+1 \leq j \leq 2s_{n-2}} Y_j^2$$

Due to the inductive hypothesis, the set $\{g^{(n-2)} = 0\}$ is a curve in the space of coordinates $Y^{(s_{n-2})}, X$. Suppose that $\{h_n^{(n-1)} = 0\}$ is not finite. Then there is some index $j \in \{1, \ldots, s_{n-2} + n\}$, such that $dim(\{g^{(n-2)} = 0\} \cap \{Z_j = m\}) = 1$, where $m = \mu_{n-1}$ or $m = 1 - \mu_{n-1}$ and Z_j is one of the variables $Y_1, \ldots, Y_{s_{n-2}}, X_1, \ldots, X_n$. It follows that there must be a definably connected component, say C, of $\{g^{(n-2)} = 0\}$, such that $C \subset \{Z_j = m\}$. We obtain a contradiction since the element μ_{n-1} is infinitesimal relative to the field $\mathbb{R}_{2n-1}$ over which the component C is defined.

We now prove that $\{g^{(n-1)} = 0\} \subset \mathbb{R}_{2n}^{n+s_{n-1}}$ consists of a finite number of points. According to the formula,

$$\begin{aligned} g_0^{(n-1)}(Y^{(s_{n-1})}, X) := (&Y_{2s_{n-2}+1}^2 + \cdots + Y_{3s_{n-2}}^2 + X_1^2 + \cdots + X_{n-1}^2) \cdot \\ &\cdot \hat{h}_0^{(n-1)}(Y_{2s_{n-2}+1}, \ldots, Y_{3s_{n-2}}, X) \cdot g^{(n-2)}(Y_{2s_{n-2}+1}, \ldots, Y_{3s_{n-2}}, X) + \\ &+ h^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \ldots, Y_{3s_{n-2}+n-1}, X_n). \end{aligned}$$

Due to the inductive hypothesis, in the $(s_{n-2} + n)$-dimensional space of coordinates $Y_{2s_{n-2}+1}, \ldots, Y_{3s_{n-2}}, X$ the set $\{g^{(n-2)} = 0\}$ is a semi-Pfaffian curve and $\{\hat{h}_0^{(n-1)} = 0\}$ is zero dimensional. Moreover $\{h^{(n-1)} = 0\}$ is zero dimensional in the $(2s_{n-2} + n)$-dimensional space of coordinates $Y_1, \ldots, Y_{2s_{n-2}}, Y_{3s_{n-2}+1}, \ldots, Y_{3s_{n-2}+n-1}, X_n$. Let

$$G_1 = \{g^{(n-2)} \cdot \hat{h}_0^{(n-1)} \cdot (Y_{2s_{n-2}+1}^2 + \cdots + Y_{3s_{n-2}}^2 + X_1^2 + \cdots + X_{n-1}^2) = 0\}$$

and $G_2 = \{h^{(n-1)} = 0\}$. By counting dimensions we get that in the $(s_{n-1} + n)$-dimensional space of all coordinates, G_1 and G_2 have dimensions $2s_{n-2} + n$ and $s_{n-2} + n - 1$ respectively. Now consider the projections $\hat{G}_1$, $\hat{G}_2$ of G_1, G_2 respectively on the subspace of coordinates $X_1, \dots, X_n$. The set $\hat{G}_1$ is a sub-Pfaffian curve (which includes the X_n-axis) having the property that for any value ω of the X_n-coordinate, the intersection $\hat{G}_1 \cap \{X_n = \omega\}$ consists of a finite number of points (inductive hypothesis). Since the set $\hat{G}_2$ is a union of hyperplanes parallel to $\{X_n = 0\}$ we deduce that the intersection $\hat{G}_1 \cap \hat{G}_2$ is zero dimensional. It follows that the set $G_1 \cap G_2$ if non empty is a transversal intersection of two planes of complementary dimensions $2s_{n-2} + n - 1$ and s_{n-2} respectively, in the s_{n-1}-dimensional space of coordinates $Y^{(3s_{n-2}+n-1)}$ and therefore zero dimensional.

Similarly one could prove that $\{g_1^{(n-1)} = 0\}$ also consists of a finite number of points. It follows that the union $\{g^{(n-1)} = 0\}$ of these two sets is also zero dimensional. $\square$

Let $G^{(i)} = \{g^{(i)} = 0\}$ and $G_\pi^{(i)} = \pi_{i+1}(G^{(i)})$. It follows from Lemma 6.2.3, that $G_\pi^{(i)}$ is a closed curve in the space $\mathbb{R}^{i+2}_{n+i+1}$ equipped with variables $X_1, \dots, X_{i+2}$. Define $H_j^{(i)} = \{h_j^{(i)} = 0\}$, $0 \le j \le i+2$, and $H^{(i)} = \{h^{(i)} = 0\}$.

Remark 6.2.4. *Suppose that $x = (x_1, \dots, x_i, x_{i+1}) \in \pi_i(H^{(i)})$, $1 \le i \le n-1$. In what follows we assume that in the description of the cell decomposition , for each i, we add the point $st_{i+1}(x)$ to the set $\Omega_s^{(i)}$ (see comments after Corollary 4.3.4). Lemma 6.2.3 states in particular that there is a finite number of such points.*

It will be convenient for the proof of the next Lemma to introduce the composite map $\rho'_n = \rho_n^{(n-1)} \circ \pi_{n-1}$ denoting the projection from the space of coordinates $Y^{(s_{n-1})}, X_1, \dots, X_n$ onto X_n.

Lemma 6.2.5. *For each point $y \in \Omega^{(n-1)}$ there exists a point $x \in \pi_{n-1}(\{g^{(n-1)} = 0\})$ such that $y = st_n(x)$.*

Proof. We will actually prove that for fixed values of variables $X_{i+2}, \dots, X_n$

$$\Omega^{(i)} = st_{i+1}(\pi_i(G^{(i)})), \quad 0 \le i \le n-1.$$

We shall proceed by induction on i. First consider the base of the induction for $i = 0$. If $f^{(0)}(X)$ is identically zero, then the set $H_0^{(0)} = \emptyset$ and $G^{(0)} = \{0, \varepsilon_0, 1, 1 - \varepsilon_0\} = \Omega_0^{(0)}$. If $\{f^{(0)}(X) = 0\}$ is a finite collection of points then $H_0^{(0)}$ is also a finite set of points, with the property that for every point in $\{f^{(0)}(X) = 0\}$, there is a point in $H_0^{(0)}$ infinitely close to it. Thus $st_1(G^{(0)}) = \Omega_0^{(0)}$.

Suppose that we have proved the lemma for all values of $i < n-1$.

We now consider the case $i = n-1$. By inductive hypothesis, for every fixed value ω of X_n-coordinate we have that

$$\Omega^{(n-2)}[\omega] = st_{n-1}(\pi_{n-2}(G^{(n-2)}(Y^{(s_{n-2})}, X_1, \dots, X_{n-1}, \omega))).$$

It follows that $\Gamma^{(n-1)} = st_{n-1}\big(G_\pi^{(n-2)}\big)$, where $G_\pi^{(n-2)} = \pi_{n-1}(G^{(n-2)})$ is a closed curve in the space $\mathbb{R}^n_{2n-1}$ equipped with coordinates $X_1, \dots, X_n$; as a result

$$\Omega_s^{(n-1)} = \mathcal{S}_n\big(st_{n-1}(G_\pi^{(n-2)})\big).$$

Claim.

$$\rho_n^{(n-1)}\big(\Omega_s^{(n-1)}\big) = \rho'_n\big(st_{n-1}(H^{(n-1)})\big). \tag{6.3}$$

We now proceed to prove this claim. The inclusion

$$\rho'_n\big(st_{n-1}(H^{(n-1)})\big) \subset \rho_n^{(n-1)}\big(\Omega_s^{(n-1)}\big)$$

follows directly from Remark 6.2.4.

Next we show that the reverse inclusion is also true. Suppose that $y = (y_1, \dots, y_n) \in \Omega_s^{(n-1)}$. We prove that $y_n \in \rho'_n\big(st_{n-1}(H^{(n-1)})\big)$.

By definition either $y \in \mathcal{E}_n(\Gamma^{(n-1)})$ or $y \in \mathcal{R}_n(\Gamma^{(n-1)})$.

(i) Let $y \in \mathcal{E}_n(\Gamma^{(n-1)})$. First we consider the case when y lies in the interior $\hat{I}^n$ of the unit cube I^n (w.l.o.g. we can assume that y is a point of local maximum of X_n-coordinate on $\Gamma^{(n-1)} \cap \hat{I}^n$).

By the inductive hypothesis, applied to the section $X_n = y_n$, there exists a point

$$(x_1, \dots, x_{n-1}, y_n) \in G_\pi^{(n-2)} \cap \{X_n = y_n\},$$

such that $st_{n-1}(x_1, \dots, x_{n-1}) = (y_1, \dots, y_{n-1})$.

Let C be a definably connected component of the curve $G_\pi^{(n-2)} \in \mathbb{R}^n_{2n-1}$ containing the point $(x_1, \dots, x_{n-1}, y_n)$.

Consider the sequence of fields $\mathbb{R}_n \subset \mathbb{R}_{\varepsilon'} \subset \mathbb{R}_{2n}$, so that $\mathbb{R}_{\varepsilon'}$ is a nonstandard extension of $\mathbb{R}_n$ containing a positive element ε' infinitesimal rel. to $\mathbb{R}_n$, and $\delta_1 \in \mathbb{R}_{2n}$ is infinitesimal rel. to $\mathbb{R}_{\varepsilon'}$, i.e., $0 < \delta_1 \ll \varepsilon' \ll \varepsilon_{n-1}$.

We claim that $C \cap \{X_n = y_n + \varepsilon'\} = \emptyset$. Indeed, suppose that this is wrong and let $v \in C \cap \{X_n = y_n + \varepsilon'\}$. Due to the inductive hypothesis and the ordering of the infinitesimals $\varepsilon' \gg \delta_1 \gg \cdots \gg \mu_{n-1}$,

$$\Gamma^{(n-1)} = st_{n-1}\big(G_\pi^{(n-2)}\big) = st_{\varepsilon'}\big(G_\pi^{(n-2)}\big),$$

where the standard part $st_{\varepsilon'}$ is taken relative to the field $\mathbb{R}_{\varepsilon'}$. Hence, $st_{\varepsilon'}(v) \in \Gamma^{(n-1)}$. It follows that there is a connected component of $\Gamma^{(n-1)}$ passing through y and intersecting $\{X_n = y_n + \varepsilon'\}$. This contradicts to the supposition that y is a point of local maximum on $\Gamma^{(n-1)}$. Thus, the claim is proved.

Then the coordinate function X_n has a point of local maximum on C, say w, such that $st_{n-1}(w_n) = y_n$. Due to the inductive hypothesis, $st_{n-1}(w) \in \Gamma^{(n-1)}$. It follows from the continuity of $g^{(n-2)}$ that $st_{n-1}(w) = y$. Obviously there exists a point w' of local maximum of X_n-coordinate on $C' \subset G^{(n-2)}$ in the space $\mathbb{R}_{2n-1}^{s_{n-2}+n}$ of coordinates $Y^{(s_{n-2})'}, X$ such that $\pi_{n-1}(C') = C$ and $\pi_{n-1}(w') = w$. In particular, $w' \in \mathcal{E}_n(G^{(n-2)})$ and $\rho_n'\big(st_{n-1}(w_n')\big) = y_n$.

In order to prove that the value $y_n \in \rho_n'\big(st_{n-1}(H^{(n-1)})\big)$, it is enough to show that $w_n = w_n' \in \rho_n'\big(st_{2n-1}(H_0^{(n-1)})\big)$; the result would follow by taking standard parts relative to $\mathbb{R}_{n-1}$. Recall that

$$\begin{aligned} h_0^{(n-1)}(Y^{(s_{n-2})}, X) &:= \big(g^{(n-2)}(Y^{(s_{n-2})}, X) - \nu_{n-1}\big)^2 + \textstyle\sum_{j=s_{n-2}+1}^{2s_{n-2}} Y_j^2 + \\ &+\Big(\Big(\tfrac{\partial g^{(n-2)}}{\partial Y_1}\Big)^2 - \nu_{n-1}^{[1]}\Delta^{(n-2)}\Big)^2 + \cdots + \Big(\Big(\tfrac{\partial g^{(n-2)}}{\partial Y_{s_{n-2}}}\Big)^2 - \nu_{n-1}^{[s_{n-2}]}\Delta^{(n-2)}\Big)^2 + \\ &+\Big(\Big(\tfrac{\partial g^{(n-2)}}{\partial X_1}\Big)^2 - \nu_{n-1}^{[s_{n-2}+1]}\Delta^{(n-2)}\Big)^2 + \cdots + \Big(\Big(\tfrac{\partial g^{(n-2)}}{\partial X_{n-1}}\Big)^2 - \nu_{n-1}^{[s_{n-2}+n-1]}\Delta^{(n-2)}\Big)^2, \end{aligned}$$

where

$$\Delta^{(n-2)} = \sum_{j=1}^{s_{n-2}} \Big(\frac{\partial g^{(n-2)}}{\partial Y_j}\Big)^2 + \sum_{1 \le j \le n} \Big(\frac{\partial g^{(n-2)}}{\partial X_j}\Big)^2.$$

Since the element ν_{n-1} is infinitesimal relative to the field $\mathbb{R}_{2n-1}$ over which the function $g^{(n-2)}$ is defined, $\{g^{(n-2)} = \nu_{n-1}\}$ is a (smooth) hypersurface in the $(s_{n-2} + n)$-dimensional space $\{Y_{s_{n-2}+1} = \cdots = Y_{2s_{n-2}} = 0\}$, on which the gradient vector of $g^{(n-2)}$ does not vanish. In addition, for each $z \in G^{(n-2)}$, there exists a point $x \in \{g^{(n-2)} = \nu_{n-1}\}$ such that $st_{2n-1}(x) = z$. Let

$$f_0^{(n-1)} := \big(g^{(n-2)} - \nu_{n-1}\big)^2 + \sum_{j=s_{n-2}+1}^{2s_{n-2}} Y_j^2 + \sum_{j=1}^{s_{n-2}} \Big(\frac{\partial g^{(n-2)}}{\partial Y_j}\Big)^2 + \sum_{j=1}^{n-1} \Big(\frac{\partial g^{(n-2)}}{\partial X_j}\Big)^2.$$

The function $f_0^{(n-1)}$ defines the (not necessarily) zero dimensional set of points on the hypersurface $\{g^{(n-2)} = \nu_{n-1}\}$ whose gradient is parallel to the X_n-axis. This is exactly the set of critical points of the projection map ρ_n' on $\{g^{(n-2)} = \nu_{n-1}\}$. By Sard's theorem, the set of critical values of ρ_n' is finite; thus $\{f_0^{(n-1)} = 0\}$ has a finite number of connected components each one lying entirely in a hyperplane orthogonal to the X_n-axis. Clearly, $\{f_0^{(n-1)} = 0\}$ contains all points of local extrema of the coordinate function X_n on the hypersurface $\{g^{(n-2)} = \nu_{n-1}\}$, that is,

$$\mathcal{E}_n(\{g^{(n-2)} = \nu_{n-1}\}) \subset \{f_0^{(n-1)} = 0\}.$$

Moreover, $st_{2n-1}(H_0^{(n-1)}) \subset st_{2n-1}(\{f_0^{(n-1)}=0\})$ and by Sard's theorem,

$$\rho_n'\big(st_{2n-1}(H_0^{(n-1)}\big) = \rho_n'\big(st_{2n-1}(\{f_0^{(n-1)}=0\}\big).$$

This implies that

$$\rho_n'\Big(st_{2n-1}\big(\mathcal{E}_n(\{g^{(n-2)}=\nu_{n-1}\})\big)\Big) \subset \rho_n'\Big(st_{2n-1}\big(H_0^{(n-1)}\big)\Big).$$

In order to complete the proof we refer to the inclusion

$$\mathcal{E}_n\big(G^{(n-2)}\big) \subset st_{2n-1}\big(\mathcal{E}_n(\{g^{(n-2)}=\nu_{n-1}\})\big),$$

which follows easily from the continuity of the function $g^{(n-2)}$ and the fact that the map st_{2n-1} is order preserving.

As a result, $w_n' \in \rho_n'\big(st_{2n-1}(H_0^{(n-1)})\big)$ and hence $y_n \in \rho_n'\big(st_{n-1}(H^{(n-1)})\big)$.

Remark. In the particular case that w' is an isolated point of the curve $G^{(n-2)}$ and for any unit $(n+s_{n-2})$-vector u, there exists a point $x_u' \in \{g^{(n-2)}=\nu_{n-1}\}$, such that $st_{2n-1}(x_u')=w'$ and the normalized gradient vector

$$\left(\frac{\partial g^{(n-2)}}{\partial Y_1},\dots,\frac{\partial g^{(n-2)}}{\partial Y_{s_{n-2}}},\frac{\partial g^{(n-2)}}{\partial X_1},\dots,\frac{\partial g^{(n-2)}}{\partial X_n}\right)\Big/\sqrt{\Delta^{(n-2)}}$$

coincides with u at x_u'. In the course of the proof of Lemma 6.2.3, we have shown that for

$$u=\left(\sqrt{\nu_{n-2}^{[1]}},\dots,\sqrt{\nu_{n-2}^{[s_{n-2}+n-1]}},\sqrt{1-\textstyle\sum_{1\le j\le s_{n-2}+n-1}\nu_{n-2}^{[j]}}\right),$$

the set of corresponding points x_u' is indeed finite.

Next we consider the case when $y \in \mathcal{E}_n(\Gamma^{(n-1)})$ lies on the boundary $\tilde{I}^n$ of the cube I^n (we may assume that y is not an isolated point of $\Gamma^{(n-1)}$, as this case can be treated as above). By inductive hypothesis $\Gamma^{(n-1)} = st_{n-1}\big(G_\pi^{(n-2)}\big)$. Since the element μ_{n-1} is infinitesimal rel. to $\mathbb{R}_{2n-1}$, the set $\pi_{n-1}(\{p_n=0\}) \cap I^n$ is infinitely (rel. to $\mathbb{R}_{n-1}$) close to $\tilde{I}^n$. It follows that $y \in \pi_{n-1}\big(st_{n-1}(H_n^{(n-1)})\big)$ and in particular, $y_n \in \rho_n'\big(st_{n-1}(H^{(n-1)})\big)$.

(ii) Let $y \in \mathcal{R}_n(\Gamma^{(n-1)})$. Consider two infinitesimals, δ' and δ'', such that

$$0<\delta_2 \ll \delta'' \ll \delta_1 \ll \delta' \ll \varepsilon_{n-1}$$

and let $\mathbb{R}_{\delta''}$ be a nonstandard extension of $\mathbb{R}_n$ containing the positive elements $\delta'', \delta_1, \delta'$.

By the definition of a ramification point, in the vicinity of y there exist two branches, say Γ' and Γ'', of $\Gamma^{(n-1)}$ such that $y \in \Gamma' \cap \Gamma''$ and there either exists a pair of points

$$\Gamma' \cap \{X_n = y_n - \delta'\}, \qquad \Gamma'' \cap \{X_n = y_n - \delta'\},$$

or exist a pair of points

$$\Gamma' \cap \{X_n = y_n + \delta'\}, \qquad \Gamma'' \cap \{X_n = y_n + \delta'\}.$$

Let, for definiteness, the first two points exist.

Define

$$y^{(1)} = (y_1^{(1)}, \dots, y_n^{(1)}) = \Gamma' \cap \{X_n = y_n - \delta'\},$$
$$y^{(2)} = (y_1^{(2)}, \dots, y_n^{(2)}) = \Gamma'' \cap \{X_n = y_n - \delta'\}.$$

Let l be the largest among the numbers $i \in \{1, \dots, n-1\}$ such that $y_i^{(1)} \neq y_i^{(2)}$. Observe that for any α_n with $y_n^{(1)} < \alpha_n < y_n$ and a pair of points $\alpha' = \Gamma' \cap \{X_n = \alpha_n\}$, $\alpha'' = \Gamma'' \cap \{X_n = \alpha_n\}$, the number l is the largest among i such that $\alpha'_i \neq \alpha''_i$.

Define $\hat{\Gamma}'$ and $\hat{\Gamma}''$ as shifts of Γ' and Γ'' respectively along X_l by δ_1 (i.e., according to the map $X_l \mapsto X_l - \delta_1$). Also define:

$$\hat{y}^{(1)} = \hat{\Gamma}' \cap \{X_n = y_n - \delta'\}, \qquad \hat{y}^{(2)} = \hat{\Gamma}'' \cap \{X_n = y_n - \delta'\},$$

$$y^{(3)} = \Gamma' \cap \{X_n = y_n - \delta''\}, \qquad y^{(4)} = \Gamma'' \cap \{X_n = y_n - \delta''\},$$
$$\hat{y}^{(3)} = \hat{\Gamma}' \cap \{X_n = y_n - \delta''\}, \qquad \hat{y}^{(4)} = \hat{\Gamma}'' \cap \{X_n = y_n - \delta''\}.$$

Note in particular, that $\hat{y}_j^{(1)} = y_j^{(2)}$ and $\hat{y}_j^{(3)} = y_j^{(4)}$ for $j = l+1, \dots, n$ (see Figure 6-5).

Observe that the curve $G_\pi^{(n-2)}$ is defined over $\mathbb{R}_{2n-1}$ (a field not containing δ', δ'' and δ_1). The set $G_\pi^{(n-2)} \cap \{y_n - \delta' < X_n < y_n - \delta''\}$ consists of some connected components. For each definably connected component C of that set, maximal and minimal values of X_n-coordinate on C, if exist, should be elements of $\mathbb{R}_{2n-1}$, thus for any such value w the distance $|w - y_n|$ is either infinitesimal relative to $\mathbb{R}_{\delta''}$ or $|w - y_n| > a > 0$ for a certain $a \in \mathbb{R}_{2n-1}$. It follows that in the interval $(y_n - \delta', y_n - \delta'')$ there are no maximal or minimal points of X_n-coordinate, therefore the component $cl(C)$ intersects $\{X_n = y_n - \delta'\}$ as well as $\{X_n = y_n - \delta''\}$.

For $1 \leq l \leq n-1$, define

$$G_l^{(n-2)} := \{g^{(n-2)}(Y^{(s_{n-2})}, X_1, \dots, X_{l-1}, X_l - \delta_1, X_{l+1}, \dots, X_n) = 0\}$$

and let

$$G_{l,\pi}^{(n-2)} = \pi_{n-1}(G_l^{(n-2)}).$$

By the inductive hypothesis (applied to the section $X_n = y_n - \delta'$), there exist points

$$x^{(2)} = (x_1^{(2)}, \ldots, x_n^{(2)}) \in G_\pi^{(n-2)} \cap \{X_n = y_n - \delta'\},$$

$$\hat{x}^{(1)} = (\hat{x}_1^{(1)}, \ldots, \hat{x}_n^{(1)}) \in G_{l,\pi}^{(n-2)} \cap \{X_n = y_n - \delta'\}$$

such that

$$st_{\delta''}(x^{(2)}) = y^{(2)}, \qquad st_{\delta''}(\hat{x}^{(1)}) = \hat{y}^{(1)}$$

where the standard part is taken relative to the field $\mathbb{R}_{\delta''}$ containing $\delta'', \delta_1, \delta'$.

Now we prove that if C is the definably connected component of $G_\pi^{(n-2)}$ containing $x^{(2)}$, then the distance between C and Γ'' is infinitesimal relative to $\mathbb{R}_{\delta''}$. Indeed, since $st_{\delta''}(x^{(2)}) = y^{(2)} \in \Gamma''$, the standard part $st_{\delta''}(C)$ intersects Γ'' at $y^{(2)}$. Suppose that there is a point $z \in C$ such that the distance $dist(z, \Gamma'')$ is not infinitesimal relative to $\mathbb{R}_{\delta''}$; then $st_{\delta''}(z) \notin \Gamma''$. Therefore, we have two curves, $st_{\delta''}(G_\pi^{(n-2)})$ and Γ'', defined over the field not containing δ' such that they intersect at a point (namely, $y^{(2)}$) with the coordinate $y_n - \delta'$, but do not coincide. This is a contradiction which proves that $dist(C, \Gamma'')$ is infinitesimal relative to $\mathbb{R}_{\delta''}$.

Similar argument shows that for a definably connected component D of the curve $G_{l,\pi}^{(n-2)} \cap \{y_n - \delta' < X_n < y_n - \delta''\}$ containing the point $\hat{x}^{(1)}$, the distance $dist(D, \hat{\Gamma}')$ is infinitesimal relative to $\mathbb{R}_{\delta''}$.

Define $x^{(4)} = C \cap \{X_n = y_n - \delta''\}$ and $\hat{x}^{(3)} = D \cap \{X_n = y_n - \delta''\}$.

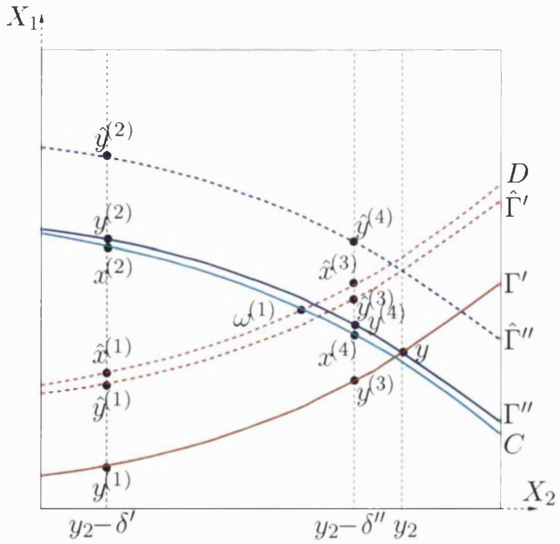


Figure 6-5: The curves $\Gamma' \cup \Gamma''$ and $C \cup D$ in the vicinity of a ramification point y of $\Gamma^{(1)}$ (here $n = 2$ and $l = 1$).

Let

$$Q_l^{(n-2)} = \{q_l^{(n-2)}(Y^{(s_{n-2})}, X) = 0\}$$

and

$$Q_{l,\delta}^{(n-2)} = \{q_l^{(n-2)}(Y_{s_{n-2}+1}, \dots, Y_{2s_{n-2}}, X_1, \dots, X_l - \delta_1, \dots, X_n) = 0\}.$$

Denote by $B_y(r)$ the sphere centered on y with arbitrarily small radius $r \in \mathbb{R}_{n-1}$. There are two cases to examine.
(a) If the intersection of $\Gamma' \cup \Gamma''$ with $B_y(r)$ is a subset of the curve $\Gamma_e^{(n-1)}$ then we let

$$C_\rho = \rho_l^{(n-2)}(C), \qquad D_\rho = \rho_l^{(n-2)}(D)$$

denote the projections of the curves C and D respectively, on the 2-plane $\{X_1 = 0, \dots, X_{l-1} = 0, X_{l+1} = 0, \dots, X_{n-1} = 0\}$ equipped with coordinates X_l, X_n (see e.g., Figure 6-4 for an example with $n = 3$, $l = 1$). Consider the points $a = \rho_l^{(n-2)}(x^{(2)}), c = \rho_l^{(n-2)}(x^{(4)}) \in C_\rho$ and $b = \rho_l^{(n-2)}(\hat{x}^{(1)}), d = \rho_l^{(n-2)}(\hat{x}^{(3)}) \in D_\rho$.
(b) If the intersection of $\Gamma' \cup \Gamma''$ with $B_y(r)$ is not a subset of the curve $\Gamma_e^{(n-1)}$ then we let

$$C_\rho = \rho_l(C), \qquad D_\rho = \rho_l(D)$$

denote the projections of the curves C and D respectively, on the space $\{X_1 = \cdots = X_{l-1} = 0\}$ equipped with coordinates $X_l, X_{l+1}, \dots, X_n$ (see e.g., Figure 6-3 for an example with $n = 3$, $l = 1$). Consider the points $a = \rho_l(x^{(2)}), c = \rho_l(x^{(4)}) \in C_\rho$ and $b = \rho_l(\hat{x}^{(1)}), d = \rho_l(\hat{x}^{(3)}) \in D_\rho$.

Due to the definition of the function $q_l^{(n-2)}$, in either case, the connected plane curves C_ρ and D_ρ are subsets of $\pi_{n-1}(Q_l^{(n-2)})$ and $\pi_{n-1}(Q_{l,\delta}^{(n-2)})$ respectively.

We assume, for definiteness, that $y_l^{(1)} < y_l^{(2)}$. It follows that $\hat{y}_l^{(1)} < y_l^{(2)}$ and $\hat{y}_l^{(3)} > y_l^{(4)}$ (since $\delta_1 \gg \delta''$). Moreover, since $\hat{\Gamma}' \cup \hat{\Gamma}''$ is obtained from $\Gamma' \cup \Gamma''$ using the δ_1-shift, the differences $y_l^{(2)} - \hat{y}_l^{(1)}$, $\hat{y}_l^{(3)} - y_l^{(4)}$ are not infinitesimal relative to $\mathbb{R}_{\delta''}$. It follows that for the corresponding points on $G_\pi^{(n-2)}$ and $G_{l,\pi}^{(n-2)}$ the similar inequalities hold, namely $\hat{x}_l^{(1)} < x_l^{(2)}$, $\hat{x}_l^{(3)} > x_l^{(4)}$.

For any value z_n with $y_n - \delta' \le z_n \le y_n - \delta''$ define the pair of points $z' = C_\rho \cap \{X_n = z_n\}$, $z'' = D_\rho \cap \{X_n = z_n\}$; then $z'_j = z''_j$ for $l+1 \le j \le n$. In particular, for $z_n = y_n - \delta'$ we get that $a_j = b_j$ and for $z_n = y_n - \delta''$ we obtain $c_j = d_j$, for $l+1 \le j \le n$. These equalities combined with the two inequalities $a_l > b_l$ and $c_l < d_l$ imply that the curve $C_\rho \cup D_\rho$ has a ramification point, say $w^{(l)}$, such that $st_{n-1}(w_n^{(l)}) = y_n$.

Let $H_{n-l}^{(n-1)} := \{h_{n-l}^{(n-1)} = 0\} = Q_l^{(n-l)} \cap Q_{l,\delta}^{(n-l)}$. It follows from the equality

$$\pi_{n-1}(H_{n-l}^{(n-1)}) = \pi_{n-1}(Q_l^{(n-l)}) \cap \pi_{n-1}(Q_{l,\delta}^{(n-l)}),$$

that $w^{(l)} \in \pi_{n-1}(H_{n-l}^{(n-1)})$. Hence $y_n \in \rho_n'\big(st_{n-1}(H^{(n-l)})\big)$.

This finishes the proof of the *Claim.*

We are now in the position to prove the inclusion

$$\Omega^{(n-1)} \subset st_n\big(\pi_{n-1}(G^{(n-1)})\big).$$

Let $y \in \Omega^{(n-1)}$. By the definition of $\Omega^{(n-1)}$, the point y is either a special point of $\Gamma^{(n-1)}$ (relative to X_n), or belongs to one of the intersections $\Gamma^{(n-1)} \cap \{X_n = z_n\}$ or $\Gamma^{(n-1)} \cap \{X_n = z_n \pm \varepsilon_{n-1}\}$, where $z = (z_1, \dots, z_n)$ is either a special point of $\Gamma^{(n-1)}$, or (by Remark 6.2.4) a point in $st_n\big(\pi_{n-1}(H^{(n-1)})\big)$.

In what follows, unless otherwise stated, we assume that the curve $\Gamma^{(n-1)}$ belongs to the space $\mathbb{R}_n^n$ equipped with coordinates $Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n$. The function $g^{(n-2)}$ stands for

$$g^{(n-2)}(Y^{(s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n),$$

while the map π_{n-1} is the projection on $Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n$ and ρ_n' is the projection on X_n. Consider the following cases.

Case 1 Let $y \in \Gamma^{(n-1)} \cap \{X_n = z_n\}$, where either $z = (z_1, \dots, z_n) \in \Omega_s^{(n-1)}$ or $z \in st_n\big(\pi_{n-1}(H^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n))\big)$.

Case 1.1 Let $z \in \Omega_s^{(n-1)}$. Then by (6.3), there exists a point

$$u \in \pi_{n-1}\big(H^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n)\big)$$

such that $st_n(u_n) = z_n$. According to the formula,

$$\begin{aligned} g_0^{(n-1)}(Y^{(s_{n-1})}, X) := {} & (Y^2_{2s_{n-2}+1} + \cdots + Y^2_{3s_{n-2}} + X_1^2 + \cdots + X_{n-1}^2) \cdot \\ & \cdot \hat{h}_0^{(n-1)}(Y_{2s_{n-2}+1}, \dots, Y_{3s_{n-2}}, X) \cdot g^{(n-2)}(Y_{2s_{n-2}+1}, \dots, Y_{3s_{n-2}}, X) + \\ & + h^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n). \end{aligned}$$

Notice that

$$L = \{Y_{2s_{n-2}+1} = \cdots = Y_{3s_{n-2}} = X_1 = \cdots = X_{n-1} = 0\} \cup \{\hat{h}_0^{(n-1)} \cdot g^{(n-2)} = 0\}$$

is a semi-Pfaffian curve in the space $\mathbb{R}_{2n}^{s_{n-2}+n}$ of coordinates $Y_{2s_{n-2}+1}, \dots, Y_{3s_{n-2}}, X$ such that, for each fixed value $\omega \in [0,1]$ of X_n-coordinate, its intersection with the hyperplane $\{X_n = \omega\}$ consists of a finite number of points. Then, for $\omega = u_n$, there exists a point

$$p \in (L \cap \{X_n = u_n\}) \subset \{G^{(n-1)}(Y^{(s_{n-1})}, X^{(n)})\},$$

such that $st_n\big(\pi_{n-1}(p)\big) = y \in \Omega^{(n-1)} \subset \Gamma^{(n-1)}$, where this time, π_{n-1} is the projection on the space $\mathbb{R}^n_n$ equipped with coordinates $X_1, \dots, X_n$ and $\Gamma^{(n-1)} = st_{n-1}\big(G_\pi^{(n-2)}\big)$ is a curve in this space. In particular, $\Omega_s^{(n-1)} \subset st_{n-1}\big(\pi_{n-1}(\{g_0^{(n-1)} = 0\})\big)$.

<u>Case 1.2</u> Let $z' \in H^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n)$ such that the standard part $st_n(\pi_{n-1}(z'_n)) = z_n$. Then this case can be proved in exactly the same way as Case 1.1, by taking $u = \pi_{n-1}(z')$.

<u>Case 2</u> Let $y \in \Gamma^{(n-1)} \cap \{X_n = z_n \pm \varepsilon_{n-1}\}$, where either $z = (z_1, \dots, z_n) \in \Omega_s^{(n-1)}$ or $z \in st_n\big(\pi_{n-1}(H^{(n-1)}(Y^{(2s_{n-2})}, Y_{3s_{n-2}+1}, \dots, Y_{3s_{n-2}+n-1}, X_n))\big)$.

This case can be proved, using the same arguments as in Case 1, with the function $g_1^{(n-1)}$ replacing $g_0^{(n-1)}$.

We have so far proved that $\Omega^{(n-1)} \subset \pi_{n-1}(G^{(n-1)})$. The reverse inclusion can be established in a similar manner. □

Lemma 6.2.6. *The cardinality of the set $\{g^{(n-1)} = 0\}$ does not exceed*

$$2^{3^{2n-2}r^2}(n!(\alpha+\beta))^{O(3^{n-1}(r+n))}.$$

Proof. Recall that f is a Pfaffian function in n variables of the order r and degree (α, β). It follows that the Pfaffian function $g^{(n-1)}$ contains $s_{n-1} + n$ variables which is less than $3^{n-2}n$ for all sufficiently large n. The components of the degree of $g^{(n-1)}$ does not exceed $16^{n-1}((n+1)!)(\alpha+\beta)$, and the order of $g^{(n-1)}$ does not exceed $3^{n-1}r$. The result follows from Khovanskii's bound (see Proposition 3.3.13). □

Lemma 6.2.5 implies in particular that the cardinality of $\Omega^{(n-1)}$ is bounded by the cardinality of $\{g^{(n-1)} = 0\}$. Since the number of cells in the described cylindrical cell decomposition $\mathcal{D}$ of I^n compatible with $Zer(f)$ is at most $O(2^n|\Omega^{(n-1)}|)$, Lemma 6.2.6 implies Theorem 6.2.1 and Theorem 6.2.2.

Remark 6.2.7. *Wilkie proved in [Wil99] that the expansion $\mathcal{R}_P$ of the ordered ring of the reals by total (unrestricted) Pfaffian functions is o-minimal. This implies in particular, that any $\mathcal{R}_P$-definable set consists of finitely many connected components each of which is $\mathcal{R}_P$-definable. It would be interesting to investigate whether it is possible (by taking into consideration Wilkie's result) to modify the construction described in this section, in order to prove an effective upper bound on the number of connected components of a $\mathcal{R}_P$-definable set, in terms of its format.*

Chapter 7

Further research

In this final Chapter we give a brief outline of a possible direction for future research work based on the methods we developed so far in this thesis.

7.1 Constructing cell decompositions which admit a CW-complex structure

First we need to introduce the notion of a finite CW-complex (see e.g., [Mas77, Hat02]).

Definition 7.1.1. *(see [Hat02, Chapter 0]) A finite CW-complex (or cell complex) is a space W constructed as follows:*

1. *Start with a discrete set W^0, the 0-cells of W;*

2. *Inductively, form the n-skeleton W^n from W^{n-1} by attaching n-cells e^n_α via continuous maps $\phi_\alpha : S^{n-1} \longrightarrow W^{n-1}$. This means that W^n is the quotient space of the disjoint union $W^{n-1} \coprod_\alpha D^n_\alpha$ of W^{n-1} with a family of copies of n-dimensional discs D^n_α under the identifications $x \sim \phi_\alpha(x)$ for $x \in S^{n-1} = bd(D^n_\alpha)$. So, as a set $W^n = W^{n-1} \coprod_\alpha e^n_\alpha$, where each e^n_α is an open n-disk.*

3. *Stop this inductive process at a finite stage, setting $W = W^n$.*

Each cell e^n_α in a CW-complex W has a continuous *characteristic map* Φ_α, the composition:

$$D^n_\alpha \hookrightarrow W^{n-1} \coprod_\alpha D^n_\alpha \xrightarrow[map]{identif.} W^n \hookrightarrow W,$$

which extends the attaching map ϕ_α and whose restriction to the interior of D^n_α is a homeomorphism onto e^n_α.

The definition of the quotient topology on $W = W^n$ implies that a set $T \subset W$ is open (or closed) if and only if $T \cap W^k$ is open (closed) in W^k for each $k \leq n$. A subcomplex of a CW-complex W is a subspace $T \subset W$ which is a union of cells of W, such that the closure of each cell in T is also contained in T.

The letters **CW** refer to the two properties satisfied by W:

1. Closure-finiteness – the closure of each cell meets finitely many other cells;

2. Weak-topology – a subset of W is closed if and only if it meets the closure of each cell in a closed set .

For instance, a graph W is a 1-dimensional CW-complex, with its edges (1-cells) being attached on its vertices (0-cells). The sphere S^n in $\mathbb{R}^{n+1}$ can be viewed as a CW-complex with only two cells: e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \longrightarrow e^0$ (in other words, we identify S^n with the quotient space $D^n/bd(D^n)$).

For more examples, as well as, some basic topological properties of CW-complexes, the interested reader can consult e.g, [Hat02, Chapter 0 and the Appendix].

In Section §4.2 we described a certain cylindrical cell decomposition $\mathcal{D}$ of the unit cube I^n compatible with a given semianalytic set $S \subset I^n$. But in general, this decomposition may fail to fulfil the following frontier condition: the frontier ∂C of any cell $C \in \mathcal{D}$ is not contained in the union of a finite number of lower dimensional cells. As a result, the outcome of our algorithm for computing cylindrical cell decompositions of sub-Pfaffian sets, presented in Section §6.1, may not always possess a CW-complex structure.

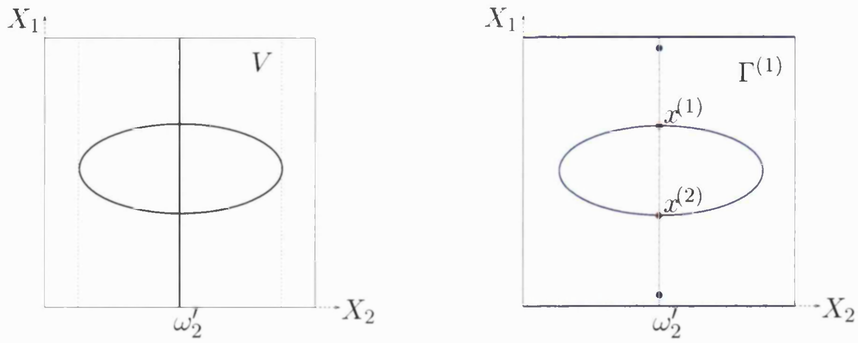


Figure 7-1: The resulting cell decomposition of I^2 compatible with V, which follows from our description, cannot be given a CW-complex structure: points $x^{(1)}$ and $x^{(2)}$ need to be added as 0-cells and the existing single 1-cell in the section $\{X_2 = \omega_2'\}$ to be subdivided accordingly.

Figure 7-1, shows one of the simplest such examples in $\mathbb{R}^2$. In this example, the intersection of V with the hyperplane $\{X_2 = \omega_2\}$ consists of a finite number of points for all $\omega_2 \in [0,1] \setminus \{\omega_2'\}$, while $V \cap \{X_2 = \omega_2'\} = I^2 \cap \{X_2 = \omega_2'\}$. In order to produce a cylindrical cell decomposition of I^2 compatible with V that admits a CW-complex structure, the points $x^{(1)}$ and $x^{(2)}$ (shown in Figure 7-1) need to be added in the list of 0-cells of this decomposition and the 1-cell belonging to the section $\{X_2 = \omega_2'\}$ to be subdivided accordingly. Going back to the description of the cylindrical cell decomposition of I^2 compatible with V (Section §4.2, case $n = 2$), we see that this can be achieved very easily by replacing in the definition of $\Omega_0^{(1)}$, the set $\widehat{\Gamma}^{(1)}$ by $\Gamma^{(1)}$, and then letting $\Omega_0^{(0)}[\omega_2] = \Omega_0^{(1)} \cap \{X_2 = \omega_2\}$ be the set which determines the 0- and 1-cells of $I^2 \cap \{X_2 = \omega_2\}$.

Next we consider a slightly more complicated example in $\mathbb{R}^3$ (see Figure 7-2). In this example, the intersection of the cube I^3 with the hyperplane $\{X_3 = \omega_3'\}$ is included in V for some value $\omega_3' \in [0,1]$ and for any neighbouring value ω_3 of ω_3', the set $V \cap \{X_3 = \omega_3\}$ is a curve homeomorphic to a circle, such that the limit set

$$V\{\omega_3'\} = cl\big(\{V \neq \omega_3'\}\big) \cap \{X_3 = \omega_3'\}$$

is an "eight-looking" curve.

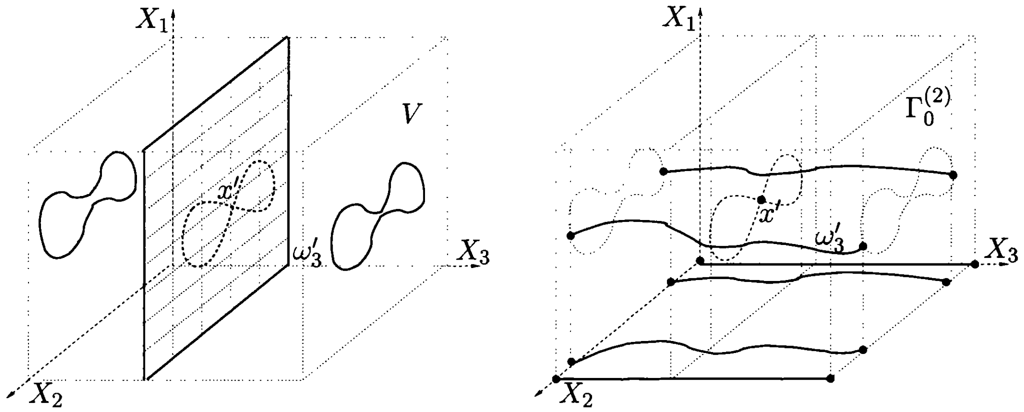


Figure 7-2: An example in which the described cylindrical cell decomposition of the unit cube I^3 compatible with V does not satisfy the extra "frontier condition".

Notice that the point x', which appears as a ramification point of the "limit curve" $V\{\omega_3'\}$ in Figure 7-2, is not contained in the curve $\Gamma^{(2)}$. So even if the open curve $\widehat{\Gamma}^{(2)}$ is replaced by its closure $\Gamma^{(2)}$ in the definition of $\Omega_0^{(2)}$, and the description of the cylindrical cells for the section $V \cap \{X_3 = \omega_3'\}$ is adjusted accordingly, we would still "miss" the 0-cell x'.

In order to produce a cylindrical cell decomposition of I^3 compatible with V, which can also be endowed with the CW-complex structure, clearly, we need to compute sets of the kind $\Omega^{(1)}$ not only for the section $V[\omega_3'] = V \cap \{X_3 = \omega_3'\}$ but also for the "limit set" $V\{\omega_3'\}$ defined above.

In general, the number of different such values ω_3' for which we would need to consider sets of the kind $\Omega^{(2)}$ for both the intersection $V[\omega_3']$ and for the "limit set" $V\{\omega_3'\}$ will always be finite.

Moreover, some sort of a backtracking procedure needs to be introduced, in order to ensure that the cylidrical arrangement of the attained cell decomposition is preserved after the addition of any new cells.

For instance, let $V \subset \mathbb{R}^3$ be as in Figure 7-3: a union of two circles, say, C_1 and C_2, such that:

- C_1 is entirely contained in the section $\{X_3 = \omega_3'\}$, for some $\omega_3' \in [0,1]$;
- for all $\omega_3 \in [0,1]$, the intersection $C_2 \cap \{X_3 = \omega_3'\}$ is finite;
- $C_1 \cap C_2$ consists of a single point $x^{(2)} = (x_1^{(2)}, x_2^{(2)}, \omega_3')$; and
- $x^{(2)}$ is not a point of local maximum of C_1 w.r.t. X_2-coordinate.

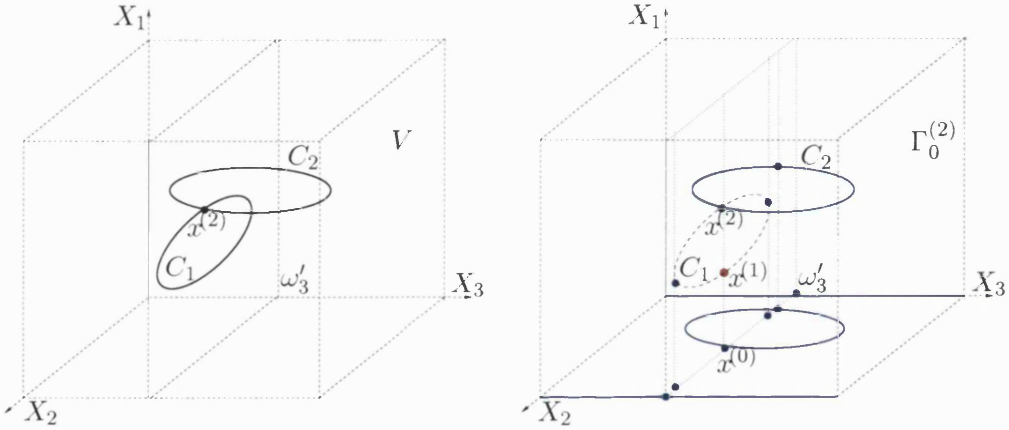


Figure 7-3: The described cylindrical cell decomposition $\mathcal{D}$ of I^3, compatible with V, does not contain the points $x^{(0)}$ and $x^{(2)}$ as 0-cells; this means that the closure of some of the 1-cells of $\mathcal{D}$ are not contained in a finite union of cells of lower dimension. Adding to $\mathcal{D}$ the points $x^{(0)}, x^{(2)}$ as 0-cells and subdividing accordingly some of its other cells is enough to turn this finer decomposition $\mathcal{D}'$ into a CW-complex. But unless we repeat this process for the point $x^{(1)}$ as well, the arrangement of the cells in $\mathcal{D}'$ will no longer be cylindrical.

It is very easy to see that for the specific example, the limit set

$$V\{\omega_3'\} = cl\big(V \cap \{X_3 \neq \omega_3'\}\big) \cap \{X_3 = \omega_3'\} = C_2 \cap \{X_3 = \omega_3'\}$$

and that the set of the kind $\Omega_0^{(1)}$ for $V\{\omega_3'\}$ is actually a subset of $\Gamma_0^{(2)} \cap \{X_3 = \omega_3'\}$. But in order to preserve the cylindricity of the induced cell decomposition of the cube $I^3 \cap \{X_3 = \omega_3'\}$ compatible with $V \cap \{X_3 = \omega_3'\}$, we need to add the point $x^{(1)}$, which belongs in the intersection

$$C_1 \cap \{X_3 = \omega_3'\} \cap \{X_2 = x_2^{(2)}\},$$

as a 0-cell, and subdivide accordingly some of the existing cells of this decomposition.

Observe that in all of the above examples, any randon perturbation of the input set V would lead to decompositions that are indeed CW-complexes. But in general, it is not clear whether performing a random linear change of the initial coordinate system at the begining of our algorithm would invariably produce a CW-complex decomposition.

For higher dimensions we expect things in general to get even more complicated. We hope to modify the described cylindrical cell decomposition $\mathcal{D}$ in such a way that the frontier ∂C of each cell $C \in \mathcal{D}$ would actually coincide with the union of a finite number of lower dimensional cells of $\mathcal{D}$.

7.2 Deducing topological results for restricted sub-Pfaffian sets

Let $S \subset I^n$ be a semi-Pfaffian set and let $W = \rho_k(S) \subset I^{n-k+1}$, $1 \leq k \leq n$, where ρ_k denotes the projection map omiting the first $k-1$ coordinates.

Let $D = D^{(0)}$ denote a cylindrical cell decomposition of I^n compatible with S. By the definition of a cylindrical cell decomposition, $D^{(0)}$ induces a cylindrical decomposition $D^{(k-1)}$ of the cube $I^{n-k+1} = \rho_k(I^n) = \{X_1 = \cdots = X_{k-1} = 0\} \subset \mathbb{R}^{n-k+1}$ compatible with W.

We will assume that the decomposition $D^{(k-1)}$ admits a CW-complex structure and that all the adjacency conditions of the cells in $D^{(k-1)}$ are known. The problem of deciding cell adjacencies is clearly reducible to that of deciding membership to closure: if C_1, C_2 are two cells of $D^{(k-1)}$ then C_1 and C_2 are adjacent only if $C_1 \subset cl(C_2)$ or $C_2 \subset cl(C_1)$. The latter problem can be solved provided we are given an oracle for deciding concistency of any system of Pfaffian equations and inequalities.

Since the decomposition $D^{(k-1)}$ is compatible with the sub-Pfaffian set W, it follows that W can be identified with a finite union of some cells of $D^{(k-1)}$. In other words, starting from a disjoint collection of cells $D^{(k-1)}$ endowed with a CW-complex structure, we can obtain W by deleting a finite number of cells from this collection. But of course, it is possible that W is not a finite CW-complex itself.

At this point, some sort of an inductive procedure needs to be introduced, which accepts as input a finite CW-complex T with a given combinatorial structure, and a list $\mathcal{U}$ consisting of some cells from T, and outputs another finite CW-complex $T_{\mathcal{U}}$ that is homotopic equivalent to the set identified with T after deleting all of its cells that appear in $\mathcal{U}$. In addition, this procedure should output all adjacency relations between cells in $T_{\mathcal{U}}$, as well as the number of its cells of different dimensions, as a function of some format of the initial input.

The existence of a CW-complex X homotopic equivalent to a sub-Pfaffian set W would immediately imply the possibility of determining some of the topological characteristics of W, such as its fundamental group or a bound on the sum of its Betti numbers.

The fundamental group $\pi_1(T)$ of a topological space T is an important topological invariant of T. In Appendix C we give the relevant definitions and discuss a method for computing the fundamental group of a CW-complex X. The fundamental group of X suffers from the serious limitation that it depends only on the 2-skeleton X^2 of X. Although higher dimensional analogues of $\pi_1(X)$ exist, in general, they are very difficult to be computed.

This leads us to consider a family of finitely generated Abelian groups, namely the homology groups of X, which are also topological invariants of X, but are easier to compute. For the exact definitions of these groups, see for example [Hat02, Chapter 2] or [Ams83, Chapter 8]. Betti numbers are ranks of homology groups and intuitively they can be thought of as expressing information regarding connectivity features of a topological space. Denote by $H_p(X)$ the p^{th} homology group of X and by $\beta_p(X)$ the p^{th} Betti number of X.

Proposition 7.2.1. *(see, e.g., [Hat02, Section 2.2]). If the number of p-dimensional cells of a CW-complex X is N_p, then the homology group $H_p(X)$ is generated by no more than N_p generators, in particular $\beta_p(X) \leq N_p$.*

We note that a single exponential in the number of variables bound for the sum of Betti numbers of a semi-Pfaffian set S was obtained in [Zel99]. Very recently a similar estimate for the sum of Betti numbers of restricted sub-Pfaffian sets defined by expressions with no negations involving atomic formulas of the kind $f \geq 0$, appeared in [GVZ02].

Appendix A

Nonstandard elementary extensions of the real ordered field

Most of the results stated in this Appendix (possibly with minor modifications) are taken from [GV96, Appendix A]. The decision to include them here was taken for the sake of completeness of this thesis. These results mainly deal with properties of subanalytic sets defined over nonstandard elementary extensions of $\mathbb{R}$.

First we recall the definition of an analytic function in an open domain of $\mathbb{R}^n$.

Definition A.1. *Let G be an open subset of $\mathbb{R}^n$. A function $f : G \longrightarrow \mathbb{R}$ is said to be analytic in G if each point $c \in G$ has a neighbourhood $U \subset G$ such that f has a power series about c over $\mathbb{R}$*

$$f(x_1, \dots, x_n) = \sum_{p_1,\dots,p_n=0}^{\infty} \alpha_{p_1 \cdots p_n}(x_1 - c_1)^{p_1} \cdots (x_n - c_n)^{p_n},$$

which is absolutely convergent for every point $x \in U$.

Lemma A.2. *Let f be an analytic function with domain $G \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^n$ a p-plane. If there exists $x \in G \cap L$ and r, $0 < r \in \mathbb{R}$, such that f vanishes in the intersection $L \cap B_x(r)$ then f vanishes in $G \cap L$.*

Proof. See Lemma A.1 in [GV96]. □

We denote by An the collection of all restricted analytic functions, that is functions of the kind $\hat{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$, for each n, defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,1]^n, \\ 0 & \text{otherwise,} \end{cases}$$

where $f : U \longrightarrow \mathbb{R}$ is an analytic function in some open neighbourhood U of the closed unit cube $[0,1]^n \subset \mathbb{R}^n$.

Let

$$\mathbb{R} = \mathbb{R}_0 \subset \mathbb{R}_1 \subset \mathbb{R}_2 \subset \cdots \subset \mathbb{R}_k \subset \cdots$$

be a sequence of real closed fields, in which the field $\mathbb{R}_{k+1}$, $k \geq 0$, contains a positive element infinitesimal relative to $\mathbb{R}_k$, and such that

$$\tilde{\mathcal{R}}^{(0)}_{An} \preceq \tilde{\mathcal{R}}^{(1)}_{An} \preceq \tilde{\mathcal{R}}^{(2)}_{An} \preceq \cdots \preceq \tilde{\mathcal{R}}^{(k)}_{An} \preceq \cdots \tag{A.1}$$

where $\tilde{\mathcal{R}}^{(k)}_{An}$ is the $\tilde{\mathcal{L}}^{(k)}_{An}$-structure $(\mathbb{R}_k, +, \cdot, -, 0, 1, \{c\}_{c \in \mathbb{R}_k}, <, An)$.

The sequence (A.1) of elementary extensions, means that the following "transfer principle" is valid: for all integers $0 \leq i < j$ and any $\tilde{\mathcal{L}}^{(i)}_{An}$-sentence Φ,

$$\tilde{\mathcal{R}}^{(i)}_{An} \models \Phi \Longleftrightarrow \tilde{\mathcal{R}}^{(j)}_{An} \models \Phi.$$

The model-completeness and o-minimality of the structures $\tilde{\mathcal{R}}^{(k)}_{An}$, $k \geq 0$, follows from the model-completeness and o-minimality of the expansion $\mathcal{R}_{An}$ of the real ordered field by restricted analytic functions [Gab68, vdD86, KPS86].

Lemma A.3. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a restricted analytic function. Denote by $f^{(1)}$ the extension of f over $\mathbb{R}_1$. Then for any point $x \in I^n \subset \mathbb{R}_1^n$, such that $B_x(r) \subset I^n$ for some $0 < r \in \mathbb{R}$, the standard part*

$$st(f^{(1)}(x)) = f(st(x))$$

and

$$st(\{f^{(1)} = \delta\}) \subset \{f = 0\},$$

where δ is an element of $\mathbb{R}_1$ infinitesimal relative to $\mathbb{R}$. If in addition there is no $y \in I^n$ and R, $0 < R \in \mathbb{R}$ such that $f(z) = 0$, $\forall z \in B_y(R)$, and $f(w) \geq 0$, $\forall w \in I^n$, then

$$st(\{f^{(1)} = \delta\}) = \{f = 0\}.$$

Proof. First, observe that any analytic function ϕ definable over $\mathbb{R}$ is continuous. Then the formula of the language $\tilde{\mathcal{L}}^{(0)}_{An}$ expressing continuity is valid for the extension

$\phi^{(1)}$ due to the transfer principle, and hence it is valid as well for (restricted) analytic functions definable over $\mathbb{R}_1$. The equality $st(f^{(1)}(x)) = f(st(x))$ and thereby the inclusion $st(\{f^{(1)} = \delta\}) \subset \{f = 0\}$ (since $st(\delta) = 0$) follows from the continuity of f and $f^{(1)}$

Now let $x \in \{f = 0\} \subset \{f^{(1)} = 0\}$. Take r, $0 < r \in \mathbb{R}$ such that $B_x(r) \subset I^n$. Consider a subanalytic set

$$D_x = \{||x - z||^2 : z \in I^n, f^{(1)}(z) = \delta\} \subset \mathbb{R}_1.$$

If it is empty then $f^{(1)}$ is less than δ everywhere on the ball $B_x(r)$ by virtue of the theorem on intermediate values of continuous functions which hold for analytic functions, hence, f vanishes everywhere on the ball $B_x(r) \cap \mathbb{R}$ (since $st(\{f^{(1)} = \delta\}) \subset \{f = 0\}$ and $st(x') = 0$, $\forall x' < \delta$) and we get a contradiction. Due to the o-minimality of $\tilde{\mathcal{R}}^{(1)}_{An}$, the set D_x is a finite union of points and intervals. Denote by u_x the minimal among these points and the endpoints of these intevals. If $st(u_x) > 0$ then the function $f^{(1)}$ on the ball $B_x(\sqrt{u_x}) \cap B_x(r)$ takes values less than δ because of the continuity of $f^{(1)}$. Let $r' = min(st(\sqrt{u_x})/2, r)$, $0 < r \in \mathbb{R}$, then f vanishes everywhere on the ball $B_x(r') \cap \mathbb{R}^n$. The obtained contradiction shows that $st(u_x) = 0$. So there exists a point $w \in \{f^{(1)} = \delta\}$ such that $||x - w||^2 = u_x$, and $st(w) = x$. $\square$

Lemma A.4. *Let a subanalytic set $W \subset \mathbb{R}^n_k$, defined by a $\tilde{\mathcal{L}}^{(k)}_{An}$-formula Ψ, be finite. Then the extension $W^{(l)} \subset \mathbb{R}^n_l$, $l > k$, of W coincides with W.*

Proof. Let $W = \{x^{(1)}, \dots, x^{(t)}\}$. Then the following formula of the language $\tilde{\mathcal{L}}^{(k)}_{An}$ is true over $\mathbb{R}_k$:

$$\bigwedge_{1 \le i \le t} \Psi(x^{(i)}) \wedge \forall X_1 \cdots \forall X_n \Big(\bigwedge_{1 \le i \le t} ((X_1, \dots, X_n) \neq x^{(i)}) \implies \neg\Psi(X_1, \dots, X_n)\Big).$$

By the transfer principle, this formula is also true over $\mathbb{R}_l$. $\square$

Definition A.5. *For an analytic function $f : G \to \mathbb{R}_k$, $G \subset \mathbb{R}^n_k$, a point $x \in G$ is called a critical point of f, if the gradient vector $(\partial f/\partial X_1, \dots, \partial f/\partial X_n)(x) = 0$. The value $f(x)$ is called a critical value of f. If x is not a critical point, the value $f(x)$ is called regular.*

Lemma A.6. *(Sard's Lemma).* *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a restricted analytic function. Then f has only finitely many critical values.*

Proof. (cf [Zel99]) Let Ξ be the set of critical points of f (which is an analytic variety). Suppose that C a connected component of Ξ and let the points x and y belong to C.

By the curve selection lemma, there is a smooth arc γ contained in C joining x to y. Since all points of this arc are in Ξ, the Mean Value Theorem shows that the restriction of f to γ is constant, so f is constant on C. The set Ξ has only a finite number of connected components, so f has only a finite number of critical values. $\square$

Corollary A.7. *For a restricted analytic function f definable over $\mathbb{R}_k$, any element $\alpha \in \mathbb{R}_l \setminus \mathbb{R}_k$ for $l > k$ cannot be a critical value of f.*

Proof. Observe that the set $\Xi_k \subset \mathbb{R}_k$ of all critical values of f is subanalytic and definable over $\mathbb{R}_k$. By Sard's Lemma, Ξ_0 consists of a finite number of points. For all subanalytic sets of the form Ξ_0, the latter statement can be expressed by a formula of the language $\tilde{\mathcal{L}}_{An}^{(0)}$. Hence, by the transfer principle the statement is true for any $k \geq 0$, that is, Ξ_k is finite. According to Lemma A.4, the extension $\Xi_k^{(l)} = \Xi_k \subset \mathbb{R}_k$, and therefore $\alpha \notin \Xi_k^{(l)}$. $\square$

Recall that for any first order formula Φ', we denote by $\mathcal{C}\Phi'$ the formula defining the topological closure (within the unit cube) of the set $\{\Phi'\}$ defined by Φ'.

Lemma A.8. *Let $W_Z = \{\Psi_Z\} \subset \mathbb{R}_k^{n+1}$ be a subanalytic set determined by an existential $\tilde{\mathcal{L}}_{An}^{(k+1)}$-formula $\Psi_Z = \exists Y_1 \cdots \exists Y_s(\Phi_Z)$ with Φ_Z quantifier free in which the atomic analytic functions are in variables $X_1, \dots, X_n, Z, Y_1, \dots, Y_s$. Let δ be an element of $\mathbb{R}_{k+1}$ infinitesimal relative to $\mathbb{R}_k$. Denote by Ψ_δ and Φ_δ the formulas which are the result of the replacement of Z by δ in Ψ_Z and Φ_Z respectively. Let $W_\delta = \{\Psi_\delta\} \subset \mathbb{R}_{k+1}^n$. Then $st_k(W_\delta) \subset \mathbb{R}_k^n$ is subanalytic and can be identified with the subset of $\mathbb{R}_k^n$ defined by the existential $\tilde{\mathcal{L}}_{An}^{(k)}$-formula*

$$\Theta \equiv \exists Y_1 \cdots \exists Y_s \Big(\mathcal{C}\big(\Phi_Z \wedge (Z > 0)\big) \wedge (Z = 0) \Big).$$

Proof. Observe that $W_\delta = \pi\{\Phi_\delta\}$, where π is the linear projection map on the subspace of coordinates $X_1, \dots, X_n$ along the coordinates $Y_1, \dots, Y_s$. We can identify the sets $\{\Phi_\delta\}$ and $\{\Phi_Z \wedge (Z = \delta)\}$.

Let us prove that

$$st_k\big(\{\Phi_Z \wedge (Z = \delta)\}\big) = cl\big(\{\Phi_Z \wedge (Z > 0)\}\big) \cap \{Z = 0\}. \tag{A.2}$$

Observe that the right-hand side of equality (A.2) is a semianalytic set.

Let $x \in st_k\big(\{\Phi_Z \wedge (Z = \delta)\}\big)$, then there exists $w \in \{\Phi_Z \wedge (Z = \delta)\}\big)$ such that $x = st_k(w)$. Hence $x \in \{Z = 0\}$. Suppose that $x \notin cl\big(\{\Phi_Z \wedge (Z > 0)\}\big)$. Then there exists an element r, $0 < r \in \mathbb{R}_k$, such that $B_x(r) \cap \{\Phi_Z \wedge (Z > 0)\} = \emptyset$. This contradicts the inclusion $w \in \{\Phi_Z \wedge (Z = \delta)\} \subset \{\Phi_Z \wedge (Z > 0)\}$.

Suppose now that

$$x \in cl\big(\{\Phi_Z \wedge (Z > 0)\}\big) \cap \{Z = 0\},$$

i.e., x belongs to the right-hand side of (A.2).

Let us prove the following claim: for any element R, $0 < R \in \mathbb{R}_k$, there exists an element α, $0 < \alpha \in \mathbb{R}_k$, such that for every β, $0 < \beta \in \mathbb{R}_k$, $\beta < \alpha$, the intersection

$$B_x(R) \cap \{\Phi_Z \wedge (Z = \beta)\}$$

is nonempty. Indeed, since the set $B_x(R) \cap \{\Phi_Z \wedge (Z > 0)\}$ is semianalytic over $\mathbb{R}_k$ (and so by the o-minimality of the structure $\tilde{\mathcal{R}}^{(k)}_{An}$, it has a finite number of connected components) there exists a connected component C of this set, such that $x \in cl(C)$. One can take as α the Z-coordinate of any point from C and the claim is proved.

It follows (with the help of the transfer principle) that for every fixed R, $0 < R \in \mathbb{R}_k$, the intersection

$$B_x(R) \cap \{\Phi_Z \wedge (Z = \delta)\} \neq \emptyset. \tag{A.3}$$

Observe that the set $D_x = \{||x - z||^2 : z \in \{\Phi_Z \wedge (Z = \delta)\} \subset \mathbb{R}_{k+1}$ is subanalytic (according to a result of Lojasiewicz on subanalytic sets in low dimensions [Loj65], D_x is actually semianalytic). By the o-minimality of $\tilde{\mathcal{R}}^{(k+1)}_{An}$, the set D_x is a finite union of points and intervals. Let $u_x \in \mathbb{R}_{k+1}$ be the minimal among these points and the endpoints of these intervals.

Suppose that $x \notin st_k\big(\{\Phi_Z \wedge (Z = \delta)\}\big)$, i.e., there does not exists $w \in \{\Phi_Z \wedge (Z = \delta)\}$ such that $st_k(w) = x$. Thus $u_x > r_1^2$ for an element $0 < r_1 \in \mathbb{R}_k$. It follows that $B_x(r_1) \cap \{\Phi_Z \wedge (Z = \delta)\} = \emptyset$. This contradicts $(A.3)$ for $R = r_1$, and equality $(A.2)$ is proved.

We have

$$st_k(W_\delta) = st_k\big(\pi(\{\Phi_Z \wedge (Z = \delta)\})\big) = \pi\big(st_k(\{\Phi_Z \wedge (Z = \delta)\})\big)$$

$$= \pi\big(cl(\{\Phi_Z \wedge (Z > 0)\} \cap \{Z = 0\})\big) = \{\Theta\},$$

and thus, $st_k(W_\delta)$ is indeed subanalytic. □

Appendix B

Cell decomposition construction: Description without infinitesimals

In Chapter 4 we have described a certain cylindrical cell decomposition of the unit cube I^n compatible with a given semianalytic set $S \subset \mathbb{R}^n$. We have actually shown that this problem can be reduced to the case when S is the zeroset $Zer(f)$ of some analytic function f in a neighbourhood G of I^n. This description in particular required the use of infinitesimal elements. This forced us to consider elementary nonstandard extensions of the real ordered field and to introduce some nonstandard techniques; as a result we had to deal with a certain amount of technical issues, which we would not have done otherwise.

It turns out that we can actually describe an alternative (although quite similar) cylindrical cell decomposition of I^n compatible with $Zer(f)$, without having to use any infinitesimal elements. The price we have to pay for this, is the introduction of even longer and more complicated formulas in the construction stage of our method.

The content of this Appendix, is an adaptation of the material from [PV03].

A description of an alternative cell decomposition

Before we begin the description of this alternative decomposition, we need one more definition.

Recall Definition 4.2.1, in which, for a subanalytic curve Δ (that is, a subanalytic set of dimension at most 1) in $\mathbb{R}^n$, we defined

- $\mathcal{E}_k(\Delta)$ to be the set of all points of local extremum of X_k-coordinate on $cl(\Delta)$;

- $\mathcal{R}_k(\Delta)$ to be the set of ramification points of $cl(\Delta)$ w.r.t. X_k-coordinate;
- $\mathcal{S}_k(\Delta) = \mathcal{E}_k(\Delta) \cup \mathcal{R}_k(\Delta)$ to be the set of special points of Δ relative to X_k-coordinate.

Definition B.1. *For a subanalytic curve $\Delta \subset \mathbb{R}^n$, define the set of frontier points $\mathcal{B}(\Delta)$ of Δ by*

$$\mathcal{B}(\Delta) := \partial\Delta = cl(\Delta) \setminus \Delta$$

and let

$$\bar{\mathcal{S}}_k(\Delta) = \mathcal{S}_k(\Delta) \cup \mathcal{B}(\Delta)$$

be the the set of extended special points of Δ relative to X_k-coordinate.

Without loss of generality we assume that $\{f = 0\} \cap I_1^n = \emptyset$. Let

$$V := (\{f = 0\} \cap I^n) \cup I_1^n.$$

We make several initial steps of the induction.

Let $n = 1$. Then let $\Lambda_s^{(0)} = \bar{\mathcal{S}}_1(V)$ and define $\Lambda_0^{(0)} = \Lambda_s^{(0)}$. For all pairs of points $x, y \in \Lambda_s^{(0)}$ consider the set $V \cap \{1/2(x+y)\}$ and denote by $\Lambda_1^{(0)}$ the union of all these sets.

Notice that if $\{f = 0\}$ is finite, then $\Lambda_1^{(0)} = \Lambda_0^{(0)}$. Each member of $\Lambda_0^{(0)}$ is a zero-dimensional cylindrical cell. A cylindrical cell decomposition $\mathcal{D}$ of I^1 compatible with V and therefore with $\{f = 0\} \cap I^1$ consists of these points and open intervals on the line between them. One can enumerate alternatively these points and intervals by successive non-negative integers j_1 in the ascending along X_1 order by assigning index $j_1 = 0$ to 0, index $j_1 = 1$ to it's neighbouring interval, and so on. Notice that $|\mathcal{D}| < 2|\Lambda_0^{(0)}|$.

Let $n = 2$. Then for every fixed value ω of X_2-coordinate define finite sets $\Lambda_0^{(0)}[\omega]$ and $\Lambda_1^{(0)}[\omega]$ as in case $n = 1$ by restricting V to $\{X_2 = \omega\}$. Let

$$\widehat{\Delta}_0^{(1)} := \bigcup_{\omega \in [0,1]} \Lambda_0^{(0)}[\omega], \qquad \widehat{\Delta}_1^{(1)} := \bigcup_{\omega \in [0,1]} \Lambda_1^{(0)}[\omega].$$

Clearly, $\widehat{\Delta}_0^{(1)}$, $\widehat{\Delta}_1^{(1)}$ are 1-dimensional (not necessarily closed) subsets of I^2. Observe that $L_1^2 \subset \widehat{\Delta}_0^{(1)} \subset \widehat{\Delta}_1^{(1)} \subset V$.

Let

$$\Lambda_s^{(1)} = \bar{\mathcal{S}}_2(\widehat{\Delta}_0^{(1)}) \cup \bar{\mathcal{S}}_2(\widehat{\Delta}_1^{(1)}).$$

For all $x = (x_1, x_2) \in \Lambda_s^{(1)}$ denote by $\Lambda_0^{(1)}$ the union of finite sets of the kind $\widehat{\Delta}_0^{(1)} \cap \{X_2 = x_2\}$. For all pairs of points $x = (x_1, x_2), y = (y_1, y_2) \in \Lambda_s^{(1)}$ denote by $\Lambda_1^{(1)}$ the union of finite sets of the kind $\widehat{\Delta}_1^{(1)} \cap \{X_2 = 1/2(x_2 + y_2)\}$ (for an example, see Figure B-1).

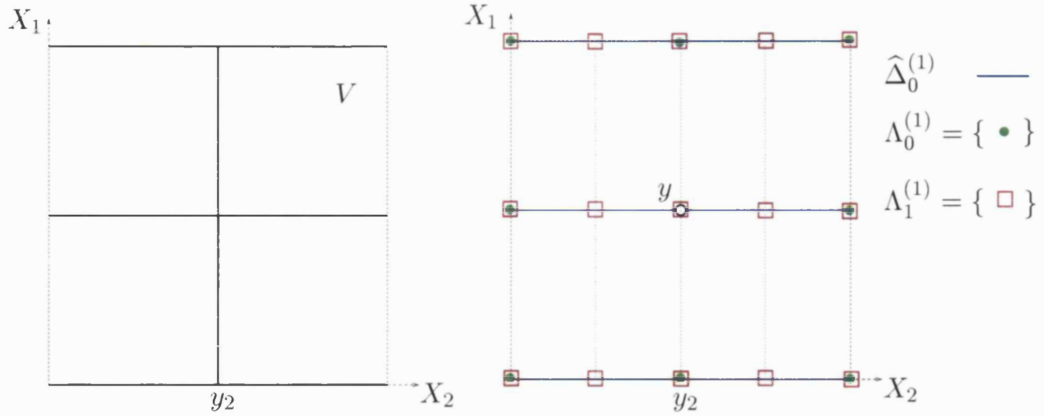


Figure B-1: The point y belongs to the frontier of the curve $\widehat{\Delta}_0^{(1)}$ and so is an extended special point of $\widehat{\Delta}_0^{(1)}$ rel. to X_2-coordinate (compare with Figure 4-4).

Let $\omega_1 < \omega_2$ be two neighbouring X_2-coordinates of points from $\Lambda_0^{(1)}$ (that is, there are no X_2-coordinates ω of points from $\Lambda_0^{(1)}$ such that $\omega_1 < \omega < \omega_2$). Then for each $\omega \in (\omega_1, \omega_2)$, the set $\Lambda_0^{(0)}[\omega] \subset \{X_2 = \omega\}$ consists of the same finite number of points. Let us enumerate these points and intervals between them, as we did in case $n = 1$, by successive non-negative integers in the ascending along X_1 order. It is clear that the set of all points having the same index for all $\omega \in (\omega_1, \omega_2)$ is an open interval of the curve $\widehat{\Delta}_0^{(1)}$, which is a one-dimensional cylindrical cell being a graph of a continuous function defined on an interval in the 1-dimensional set $L_1^2(0)$. The set of all intervals having the same index for all $\omega \in (\omega_1, \omega_2)$ is an open 2-dimensional cylindrical cell being the set of points strictly between the non-intersecting graphs of two continuous functions defined on an interval in $L_1^2(0)$.

Now we can describe all zero-, one-, and two-dimensional cells of the cylindrical decomposition of I^2 that is compatible with V. Enumerate each cell by a 2-multi-index (j_1, j_2) in a following way. Index j_2 enumerates (by successive non-negative integers starting from zero) alternatively points in $\Lambda_0^{(1)} \cap L_1^2(0)$ and intervals between these points on $L_1^2(0)$, in the ascending along X_2 order. For a fixed value of j_2, index j_1 enumerates points in $\Lambda_0^{(0)}[\omega] \subset \{X_2 = \omega\}$ and intervals between them (as in case $n = 1$), where ω is either the X_2-coordinate of the point in $\Lambda_0^{(1)} \cap L_1^2(0)$ having index j_2, or the X_2-coordinate of a point in the interval between two neighbouring points of $\Lambda_0^{(1)} \cap L_1^2(0)$ having index j_2.

It is easy to see that the defined family of the cylindrical cells is a cylindrical cell decomposition of I^2 compatible with V and therefore with $\{f = 0\} \cap I^2$. A cell having index (i, j) is cylindrical over the cell with index $(0, j)$ that belongs in the decomposition induced on $L_1^2(0)$. Observe that the number of cells in this decomposition is $O(|\Lambda_0^{(1)}|)$.

We proceed to the description of a general induction step.

For every fixed value ω of X_n-coordinate finite sets of points of the kind $\Lambda_0^{(n-2)}[\omega]$ and $\Lambda_1^{(n-2)}[\omega]$ can be defined by applying the inductive hypothesis to the intersection $V \cap \{X_n = \omega\}$. An important property of these sets, is that there are formulas (with quantifiers) $\Phi_0^{(n-2)}(X_1, \dots, X_{n-1}, X_n)$ and $\Phi_1^{(n-2)}(X_1, \dots, X_{n-1}, X_n)$ having free variables $X_1, \dots, X_n$ and not depending on ω, such that the replacement of the variable X_n by ω gives formulas $\Phi_0^{(n-2)}(X_1, \dots, X_{n-1}, \omega)$ and $\Phi_1^{(n-2)}(X_1, \dots, X_{n-1}, \omega)$ in free variables $X_1, \dots, X_{n-1}$ defining the sets $\Lambda_0^{(n-2)}[\omega]$ and $\Lambda_1^{(n-2)}[\omega]$ respectively, for the section $\{X_n = \omega\}$. Let

$$\widehat{\Delta}_0^{(n-1)} := \{\Phi_0^{(n-2)}(X_1, \dots, X_{n-1}, X_n)\}, \quad \widehat{\Delta}_1^{(n-1)} := \{\Phi_1^{(n-2)}(X_1, \dots, X_{n-1}, X_n)\}.$$

Clearly, $\widehat{\Delta}_0^{(n-1)}$, $\widehat{\Delta}_1^{(n-1)}$ are 1-dimensional (not necessarily closed) subsets of I^n. Observe that $L_{n-1}^n \subset \widehat{\Delta}_0^{(n-1)} \subset \widehat{\Delta}_1^{(n-1)} \subset V$.

Moreover, for any $k = 2, \dots, n-1$ and for $* \in \{0, 1\}$, by the actual definition of the curve $\widehat{\Delta}_*^{(n-1)}$, we have the inclusions $\rho_k(\widehat{\Delta}_*^{(n-1)}) \subset \widehat{\Delta}_*^{(n-1)}$, where ρ_k denotes the projection on the subspace of coordinates $X_k, X_{k+1} \dots, X_n$.

Let

$$\Lambda_s^{(n-1)} = \bar{\mathcal{S}}_n(\widehat{\Delta}_0^{(n-1)}) \cup \bar{\mathcal{S}}_n(\widehat{\Delta}_1^{(n-1)}).$$

For all points $x = (x_1, \dots, x_n) \in \Lambda_s^{(n-1)}$ denote by $\Lambda_0^{(n-1)}$ the union of finite sets of the kind $\widehat{\Delta}_0^{(n-1)} \cap \{X_n = x_n\}$. For all pairs of points $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \Lambda_s^{(n-1)}$ denote by $\Lambda_1^{(n-1)}$ the union of finite sets of the kind $\widehat{\Delta}_1^{(n-1)} \cap \{X_n = 1/2(x_n + y_n)\}$.

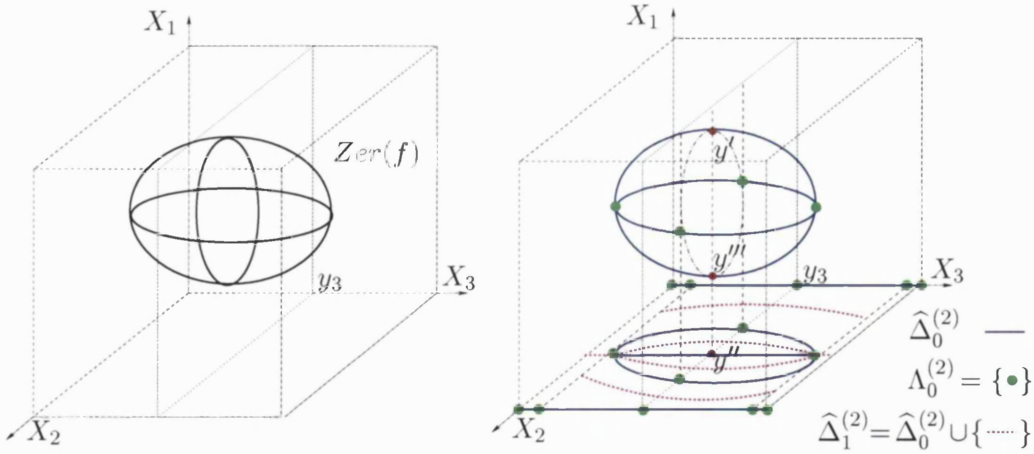


Figure B-2: The points y', y'', y''' belong to the set of frontier points of the curve $\widehat{\Delta}_0^{(2)}$, and thus, they are extended special points of $\widehat{\Delta}_0^{(2)}$ rel. to X_3-coordinate (compare with Figure 4-8). Note that here $Zer(f)$ is a curve in I^3 (not a sphere).

Let the index j_n enumerate in the ascending along X_n order, alternatively points in $\Lambda_0^{(n-1)} \cap L_{n-1}^n(0)$ and intervals between these points on $L_{n-1}^n(0)$. Let $\omega_1 < \omega_2$ be two neighbouring X_n-coordinates of points from $\Lambda_0^{(n-1)}$.

Assume that the interval (ω_1, ω_2) is indexed by j_n and a point $\omega \in (\omega_1, \omega_2)$. It follows from the inductive hypothesis that there is a certain cylindrical cell decomposition of the intersection $I_n^n[\omega] = I^n \cap \{X_n = \omega\}$ compatible with $V \cap \{X_n = \omega\}$ and all cells are enumerated by $(n-1)$-multi-indices. In fact, for all $\omega \in (\omega_1, \omega_2)$ the sets of multi-indices coincide. Moreover, any fixed multi-index corresponds to cells of the same dimension and the union of all p-cells for $p = 0, 1, \ldots, n-1$ having the same multi-index $(j_1, \ldots, j_{n-1})$ for all $\omega \in (\omega_1, \omega_2)$ is a cylindrical $(p+1)$-cell to which we will assign multi-index $(j_1, \ldots, j_{n-1}, j_n)$.

Let ω be the X_n-coordinate of the point in $\Lambda_0^{(n-1)} \cap L_{n-1}^n(0)$ having index j_n. By the inductive hypothesis there is a cylindrical cell decomposition of $I_n^n[\omega]$. All cells of this decomposition are also the elements of a cell decomposition of I^n. If a cell in $I_n^n[\omega]$ has a multi-index $(j_1, \ldots, j_{n-1})$, then considering it as a cell in I^n we assign to it the multi-index $(j_1, \ldots, j_{n-1}, j_n)$.

Observe that its total number of cells is $O(|\Lambda_0^{(n-1)}|)$.

Lemma B.2. *The cylindrical cell decomposition $\mathcal{D}$ described above is well defined and is compatible with V, and therefore, with $\{f = 0\} \cap I^n$.*

Proof. The proof of this lemma follows closely the proofs given in Section §4.3. $\square$

Constructing the described cylindrical cell decomposition

The first step towards constructing the described cylindrical cell decomposition is to build existential formulas defining the sets $\Lambda^{(i)}$, $0 \leq i \leq n-1$.

Define recursively the sequence of integers $s_0, \ldots, s_{n-1}$ by setting $s_0 = 2$ and $s_{i+1} = 5s_i + 2i + 4$ for $0 \leq i \leq n-2$. Introduce new variables $Y_1, \ldots, Y_{s_{n-1}}, Z$.

Let $D_i := (X_i \geq 0) \wedge (X_i \leq 1)$, so that $\{\bigwedge_{1 \leq i \leq k} D_i\} = I^k$. Denote $T^{(m)} = (T_1, \ldots, T_m)$, the m-tuple of variables T_i, $i \leq m$.

Let $X = X^{(n)}$. We now introduce formulas $G_0^{(i)}, G_1^{(i)}, G^{(i)}$ by induction on i. In steps $i = 0, 1$ we include comments explaining all non-trivial stages of this construction. Note that at a step i we treat $X_{i+2}, \ldots, X_n$ as parameters of these formulas.

Step i = 0.

$f^{(0)}(X) := \left(f(X)\right)^2$

In case when $f \not\equiv 0$ the equation $f^{(0)}(X) = 0$ defines the set $\Lambda_0^{(0)}$ and possibly some points outside $[0, 1]$.

$h_Z^{(0)}(X, Z) := (f^{(0)}(X) - Z)^2$
The points satisfying $f^{(0)}(X) = 0$ are perturbed by Z.

$H_Z^{(0)} := (h_Z^{(0)} = 0) \vee (X_1(X_1 - 1) = 0)$

$\Theta_e^{(0)} := \mathcal{C}(H_Z^{(0)} \wedge (Z > 0)) \wedge (Z = 0)$
This formula defines the limits of perturbed points as $Z \to +0$, i.e, the set $\Lambda_0^{(0)}$ and possibly some points outside $[0, 1]$.

$\widehat{G}_0^{(0)}(Y^{(2)}, X) := \Theta_e^{(0)} \wedge (Y_1 = Y_2 = 0) \wedge D_1$
Defining the set $\Lambda_0^{(0)}$ and the (parametric) curve $\widehat{\Delta}_0^{(1)}$ as projections along variables Y_1, Y_2.

$$\widehat{G}_1^{(0)}(Y^{(2)}, X) := (X_1(X_1 - 1)f^{(0)}(X) = 0) \wedge (X_1 = 1/2(Y_1 + Y_2)) \wedge \\ \wedge \Theta_e^{(0)}(Y_1, X_2, \ldots, X_n) \wedge \Theta_e^{(0)}(Y_2, X_2, \ldots, X_n) \wedge D_1$$
Defining the set $\Lambda_1^{(0)}$ and the (parametric) curve $\widehat{\Delta}_1^{(1)}$ as projections along variables Y_1, Y_2.

Step i = 1.

$$G_*^{(0)}(Y^{(2)}, X) := \mathcal{C}(\widehat{G}_*^{(0)}(Y^{(2)}, X)) \wedge D_2(X) \\ \equiv \bigvee_{1 \le l \le M_1} ((f_{l*}^{(0)}(Y^{(2)}, X) = 0) \wedge (g_{l*}^{(0)}(Y^{(2)}, X) > 0)) \qquad \text{for } * \in \{0, 1\}.$$
Representing each $G_*^{(0)}(Y^{(2)}, X)$ as a Boolean combination of atomic equations and inequalities (recall that $\mathcal{C}\Phi$ denotes the formula defining the topological closure of the set $\{\Phi\}$ defined by Φ).

For each l, $1 \le l \le M_0$, define :
$h_{l,Z,*}^{(1)}(Y^{(4)}, X, Z) := (f_{l*}^{(0)}(Y^{(2)}, X) - Z)^2 + \left(\frac{\partial f_{l*}^{(0)}}{\partial X_1}\right)^2 + \left(\frac{\partial f_{l*}^{(0)}}{\partial Y_1}\right)^2 + \left(\frac{\partial f_{l*}^{(0)}}{\partial Y_2}\right)^2 + Y_3^2 + Y_4^2$
For small positive values of Z the equation $f_{l*}^{(0)}(X) = Z$ defines a smooth hypersurface. Then $h_{l,Z,*}^{(1)}(Y^{(4)}, X, Z) = 0$ defines the set of all critical points of the coordinate function X_2 on this hypersurface. The purpose of introducing variables Y_3, Y_4 will be explained below.

$H_{Z*}^{(1)} := \bigvee_{1 \le l \le M_1} ((h_{l,Z,*}^{(1)} = 0) \wedge (g_{l*}^{(0)} > 0))$
Collecting together the critical points on $f_{l*}^{(0)}(X) = Z$ for all l, $1 \le l \le M_0$ and selecting the ones which are relevant. Note that for small values of $Z > 0$ all points of local extrema of the coordinate function X_2 on $\{G_*^{(1)}(Y^{(2)}, X)\}$ are close to corresponding critical points.

$\Theta_{e*}^{(1)}(Y^{(4)}, X) := \mathcal{C}(H_{Z*}^{(1)} \wedge (Z > 0)) \wedge (Z = 0)$
Passing to limit as $Z \to +0$ produces a finite (parameterized) set of points on $\{G_*^{(0)}\}$ which includes all points of local extrema of X_2 on $\{G_*^{(0)}\}$. The projection of the set

$\{\Theta^{(1)}_{e*}(Y^{(4)}, X)\}$ along variables Y_1, Y_2, Y_3, Y_4 contains all points of local extrema of X_2 on the curve $cl(\widehat{\Delta}^{(1)}_*)$.

$\Theta^{(1)}_{\partial *}(Y^{(4)}, X) := \partial(\widehat{G}^{(1)}_*(Y^{(2)}, X)) \wedge (Y_3 = Y_4 = 0)$
Defining a finite set of frontier points of $\{\widehat{G}^{(1)}_*(Y^{(2)}, X)\}$ (by abuse of notation, we denote by $\partial\Phi$ the formula defining the set of frontier points of $\{\Phi\}$). The projection of the set $\{\Theta^{(1)}_{\partial *}(Y^{(4)}, X)\}$ along variables Y_1, Y_2, Y_3, Y_4 contains $\mathcal{B}(\widehat{\Delta}^{(1)}_*)$.

$G^{(0)}_{1*} := G^{(1)}_*(Y^{(2)}, X_1 - Z, X_2, \dots, X_n)$
This defines a curve obtained from $\{G^{(1)}_*\}$ by shifting it along the coordinate axis X_1 by Z.

$Q^{(1)}_{Z*}(Y^{(4)}, X_1 - Z, X) := G^{(1)}_{1*}(Y^{(2)}, X_1 - Z, X_2, \dots, X_n) \wedge G^{(1)}_*(Y_3, Y_4, X)$
Intersecting the projection onto the X coordinates of $\{G^{(1)}_*\}$ with the projection of its shift produces a finite (parameterized) subset of $cl(\widehat{\Delta}^{(1)}_*)$. Note that we need two additional variables Y_3, Y_4. Observe that for a small value $|Z|$ each ramification point of $cl(\widehat{\Delta}^{(1)}_*)$ is close to the projection along Y_1, Y_2, Y_3, Y_4 of some of the points from $\{Q^{(1)}_{Z*}\}$.

$\Theta^{(1)}_{r*}(Y^{(4)}, X_1 - Z, X) := \mathcal{C}(Q^{(1)}_{Z*} \wedge (Z > 0)) \wedge (Z = 0)$
Passing to limit as $Z \to +0$ produces a finite (parameterized) set of points on $\{G^{(1)}_*\}$ such that its projection along variables Y_1, Y_2, Y_3, Y_4, X_1 contains all X_2-coordinates of ramification points of $cl(\widehat{\Delta}^{(1)}_*)$.

$\Theta^{(1)}_s(Y^{(4)}, X) := \Theta^{(1)}_{e0} \vee \Theta^{(1)}_{\partial 0} \vee \Theta^{(1)}_{r0} \vee \Theta^{(1)}_{e1} \vee \Theta^{(1)}_{\partial 1} \vee \Theta^{(1)}_{r1}$
Defining a set whose projection along variables $Y_1, \dots, Y_4, X_1$ is a finite set containing all X_2-coordinates of points from $\bar{\mathcal{S}}_2(\widehat{\Delta}^{(1)}_0) \cup \bar{\mathcal{S}}_2(\widehat{\Delta}^{(1)}_1)$.

$\widehat{G}^{(1)}_0(Y^{(14)}, X) := \widehat{G}^{(0)}_0(Y_5, Y_6, X) \wedge \Theta^{(1)}_s(Y^{(4)}, Y_7, X_2, \dots, X_n) \wedge (Y_8 = \dots = Y_{14} = 0) \wedge D_2$
Defining a set whose projection along variables $Y_1, \dots, Y_{14}$ contains $\Lambda^{(1)}_0$. Note that in the expression $\Theta^{(1)}_s(Y^{(4)}, Y_7, X_2, \dots, X_n)$ variable Y_7 stands for X_1, while in the expression $\widehat{G}^{(0)}_0(Y_5, Y_6, X)$ variables Y_5, Y_6 stand for Y_1, Y_2 respectively. For any fixed values of parameters $X_3, \dots, X_n$ the set $\Theta^{(1)}_s$ is finite and therefore the set $\{\widehat{G}^{(0)}_0 \wedge \Theta^{(1)}_s\}$ reduces to an intersection of two finite unions of affine subspaces of complementary dimensions in 8-dimensional space. It follows that $\widehat{G}^{(1)}_0(Y^{(14)}, X)$ is finite.

$$\widehat{G}^{(1)}_1(Y^{(14)}, X) := \widehat{G}^{(0)}(Y_9, Y_{10}, X) \wedge \big(X_2 = 1/2(Y_{12} + Y_{14}) \wedge D_2 \wedge \\ \wedge \Theta^{(1)}_s(Y^{(4)}, Y_{11}, Y_{12}, X_3, \dots, X_n) \wedge \Theta^{(1)}_s(Y_5, \dots, Y_8, Y_{13}, Y_{14}, X_3, \dots, X_n)\big)$$
The set defined by the formula $\Theta^{(1)}_s(Y^{(4)}, Y_{11}, Y_{12}, X_3, \dots, X_n)$, say T_1, is finite in the space of coordinates $Y^{(4)}, Y_{11}, Y_{12}, X_3, \dots, X_n$. Similarly, the set defined by the formula $\Theta^{(1)}_s(Y_5, \dots, Y_8, Y_{13}, Y_{14}, X_3, \dots, X_n)$, say T_2, is finite in the space of coordinates $Y_5, \dots, Y_8, Y_{13}, Y_{14}, X_3, \dots, X_n$. So for any fixed values of variables Y_9, Y_{10}, X_1, X_2 the

set $T_\Theta = T_1 \cap T_2$ reduces to a tranversal intersection of sets of complementary dimension and is therefore finite. Thus, in the space of all coordinates $Y^{(14)}, X$, the set T_Θ is a finite union of 4-dimensional hyperplanes along the variables Y_9, Y_{10}, X_1, X_2. The set $\{X_2 = 1/2(Y_{12} + Y_{14})\}$ is a 2-plane in the space of coordinates X_2, Y_{12}, Y_{14} and hence a 15-dimensional set in the space of all coordinates, whose intersection with T_Θ is a finite union of 3-dimensional hyperplanes along the variables Y_9, Y_{10}, X_1 (for every pair of fixed values for Y_{12}, Y_{14}, the value of X_2 is also fixed). On the other hand the formula $\widehat{G}^{(0)}(Y_9, Y_{10}, X)$ defines a 0-dimensional set in the space with coordinates Y_9, Y_{10}, X_1 and hence a 13-dimensional set in the space of all coordinates $Y^{(14)}, X$. It follows that the set defined by $\widehat{G}_1^{(1)}(Y^{(14)}, X)$ is a transversal intersection of two finite unions of hyperplanes of complementary dimensions and is therefore finite. Its projection along variables $Y_1, \ldots, Y_{14}$ contains $\Lambda_1^{(1)}$.

General step. Assume that on step i, $i \leq n-2$, the expressions

$$\widehat{G}_0^{(i)}(Y^{(s_i)}, X), \qquad \widehat{G}_1^{(i)}(Y^{(s_i)}, X)$$

were defined. The interpretations of the following formulas are analogous to those provided in **step 1**.

Step $(\mathbf{i+1})$.

$$G_*^{(i)}(Y^{(s_i)}, X) := \mathcal{C}(\widehat{G}_*^{(i)}(Y^{(s_i)}, X)) \wedge D_{i+2}(X)$$
$$\equiv \bigvee_{1 \leq l \leq M_i} \left((f_{l*}^{(i)}(Y^{(s_i)}, X) = 0) \wedge (g_{l*}^{(i)}(Y^{(s_i)}, X) > 0) \right) \quad \text{for } * \in \{0,1\}$$

For each l, $1 \leq l \leq M_i$, define:

$$h_{l,Z,*}^{(i+1)}(Y^{(2s_i)}, X, Z) := \left(f_{l*}^{(i)} - Z\right)^2 + \sum_{1 \leq j \leq i+1} \left(\frac{\partial f_{l*}^{(i)}}{\partial X_j} \right)^2 + $$
$$+ \sum_{1 \leq j \leq s_i} \left(\frac{\partial f_{l*}^{(i)}}{\partial Y_j} \right)^2 + \sum_{s_i+1 \leq j \leq 2s_i} \left(Y_j \right)^2$$

$$H_{Z*}^{(i+1)} := \bigvee_{1 \leq l \leq M_i} \left((h_{l,Z,*}^{(i)} = 0) \wedge (g_{l*}^{(i)} > 0) \right)$$

$$\Theta_{c*}^{(i+1)} := \mathcal{C}(H_{Z*}^{(i)} \wedge (Z > 0)) \wedge (Z = 0)$$

$$\Theta_{\partial*}^{(i+1)} := \partial(\widehat{G}_*^{(i)}(Y^{(s_i)}, X)) \wedge (Y_{s_i+1} = \cdots = Y_{2s_i} = 0)$$

$$G_{1*}^{(i)} := G_*^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1, \ldots, X_i, X_{i+1} - Z, X_{i+2}, \ldots, X_n)$$

$$G_{2*}^{(i)} := G_*^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1, \ldots, X_{i-1}, X_i - Z, X_{i+1}, \ldots, X_n)$$

......

$$G_{j*}^{(i)} := G_*^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1, \ldots, X_{i+1-j}, X_{i+2-j} - Z, X_{i+3-j}, \ldots, X_n)$$

......

$$G_{i+1*}^{(i)} := G_*^{(i)}(Y_{1+s_i}, \ldots, Y_{2s_i}, X_1 - Z, X_2, \ldots, X_n)$$

$$Q_{1,Z,*}^{(i+1)}(Y^{(2s_i)}, X_{i+1} - Z, X) := G_{1*}^{(i)} \wedge G_*^{(i)}$$

$Q^{(i+1)}_{2,Z,*}(Y^{(2s_i)}, X_i - Z, X) := G^{(i)}_{2*} \wedge G^{(i)}_{*}$

......

$Q^{(i+1)}_{j,Z,*}(Y^{(2s_i)}, X_{i+2-j} - Z, X) := G^{(i)}_{j*} \wedge G^{(i)}_{*}$

......

$Q^{(i+1)}_{i+1,Z,*}(Y^{(2s_i)}, X_1 - Z, X) := G^{(i)}_{i+1*} \wedge G^{(i)}_{*}$

$Q^{(i+1)}_{Z*}(Y^{(2s_i)}, X_1 - Z, \ldots, X_{i+1} - Z, X) := \bigvee_{1 \le j \le i+1} (Q^{(i+1)}_{j,Z,*})$

$\Theta^{(i+1)}_{r*} := \mathcal{C}(Q^{(i+1)}_{Z*} \wedge (Z > 0)) \wedge (Z = 0)$

$\Theta^{(i+1)}_{s}(Y^{(2s_i)}, X) := \Theta^{(i+1)}_{e0} \vee \Theta^{(i+1)}_{\partial 0} \vee \Theta^{(i+1)}_{r0} \vee \Theta^{(i+1)}_{e1} \vee \Theta^{(i+1)}_{\partial 1} \vee \Theta^{(i+1)}_{r1}$

$$\begin{aligned}\widehat{G}^{(i+1)}_0(Y^{(s_{i+1})}, X) := {} & \widehat{G}^{(i)}_0(Y_{1+2s_i}, \ldots, Y_{3s_i}, X) \wedge (Y_{3s_i+i+2} = \cdots = Y_{s_{i+1}} = 0) \wedge \\ & \wedge D_{i+2} \wedge \Theta^{(i+1)}_s(Y^{(2s_i)}, Y_{3s_i+1}, \ldots, Y_{3s_i+i+1}, X_{i+2}, \ldots, X_n)\end{aligned}$$

$$\begin{aligned}\widehat{G}^{(i+1)}_1(Y^{(s_{i+1})}, X) := {} & \widehat{G}^{(i)}_1(Y_{1+4s_i}, \ldots, Y_{5s_i}, X) \wedge \big(X_{i+2} = 1/2(Y_{5s_i+i+2} + Y_{5s_i+2i+4})\big) \wedge \\ & \wedge D_{i+2} \wedge \Theta^{(i)}_s(Y^{(2s_i)}, Y_{5s_i+1}, \ldots, Y_{5s_i+i+2}, X_{i+3}, \ldots, X_n) \wedge \\ & \wedge \Theta^{(i)}_s(Y_{2s_i+1}, \ldots, Y_{4s_i}, Y_{5s_i+i+3}, \ldots, Y_{5s_i+2i+4}, X_{i+3}, \ldots, X_n)\end{aligned}$$

$\widehat{G}^{(i+1)}\big(Y^{(s_{i+1})}, X\big) := \widehat{G}^{(i+1)}_0 \vee \widehat{G}^{(i+1)}_1$

End of the general step.

For each k, $0 \le k \le n-1$, let $\pi_k : \mathbb{R}^{s_k+k+1} \to \mathbb{R}^{k+1}$ be the projection map along $Y^{(s_k)}$ onto the subspace with coordinates $X_1, \ldots, X_{k+1}$. Consider a vector $(\omega_{k+2}, \ldots, \omega_n)$, such that $0 \le \omega_j \le 1$ for all $j = k+2, \ldots, n$. For any $* \in \{0,1\}$, let $\Lambda^{(k)}_*[\omega_{k+2}, \ldots, \omega_n]$ denote the set $\Lambda^{(k)}_*$ for $V \cap \{X_{k+2} = \omega_{k+2}, \ldots, X_n = \omega_n\}$ in the cube I^{n-k+1} identified with $I^n \cap \{X_{k+2} = \omega_{k+2}, \ldots, X_n = \omega_n\}$.

Lemma B.3. *For any $* \in \{0,1\}$, the projection*

$$\pi_k(\{\widehat{G}^{(k)}_*(Y^{(s_k)}, X)\} \cap \{X_{k+2} = \omega_{k+2}, \ldots, X_n = \omega_n\})$$

is a finite set of points containing $\Lambda^{(k)}_*[\omega_{k+2}, \ldots, \omega_n]$.

Proof. The proof follows very closely the proofs of a series of Lemmata, appearing in Section §5.1.2, which establish the corresponding result for the set $\Omega^{(k)}$. For $k = 2$, it is actually contained in the comments following the definition of formulas $\widehat{G}^{(1)}_*$ (see **step i = 1** above). □

Once we have constructed existential formulas defining the sets $\Lambda^{(k)}$, $0 \le k \le n-1$, we can proceed to define the cylindrical cell decomposition described above in exactly the same manner as we did in Section §5.2.

In the specific case when the input semianalytic set $S \subset \mathbb{R}^n$ is actually semi-Pfaffian with format (N, α, β, r, n), we can prove the following result.

Lemma B.4. *For each* $k \in \{0, \ldots, n-1\}$ *and* $* \in \{0,1\}$, *the format of the sets defined by* $G_*^{(k)}$ *is* $(N_k, \alpha, \beta_k, r_k, m_k)$, *where*

$$N_k = (\alpha + \beta N)^{(r+n)^{O(k)} 2^{O(k^2)}}, \beta_k = (\alpha + \beta N)^{(r+n)^{O(k)} 2^{O(k^2)}}, r_k = r5^{k-1}, m_k = O(n5^k).$$

Proof. This is very similar to the proof of Lemma 6.1.2. □

Following closely the construction of Section §6.1, we can build a two-stage algorithm for producing the described cylindrical cell decomposition. It turns out that the complexity estimate of the algorithm, as well as the number of cells in this decomposition and the estimates for the components of the format of each cylindrical cell are the same as those appearing in Theorem §6.1.1.

Remark B.5. *This alternative cylindrical cell decomposition does not seem likely to support a constructive method (similar to the one presented in Section §6.2) for establishing an improved upper bound on its number of cells.*

Appendix C

Computing the fundamental group of a CW-complex

Let X be a finite CW-complex. In this Appendix we introduce the fundamental group $\pi_1(X)$ of X, an algebraic construction which dates back to the work of Poincare, and present a method for computing it. It is not difficult to show that $\pi_1(X)$ is a topological invariant of X: homeomorphic spaces have isomorphic fundamental groups. This allows for the possibility to prove that two spaces are not homeomorphic by showing that their fundamental groups are not isomorphic. Good reference texts for this subject include the books [Mas77, Ams83, Hat02]).

Definition C.1. *Let $f, g : X \longrightarrow Y$ be maps. Then f is homotopic to g relative to a subset A of X if there exists a map $F : X \times I \longrightarrow Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, for all points $x \in X$ and $F(a,t) = f(a)$ for all $a \in A$ and for all $t \in I$.*

By a *loop* in a space X we mean a map $\alpha : I \longrightarrow X$ such that $\alpha(0) = \alpha(1)$; we say that the loop is based at the point $\alpha(0)$. If α and β are two loops based at the same point v of X, we define the product $\alpha \cdot \beta$ to be the loop given by the formula

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2, \\ \beta(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Saying that the loop α can be continuously deformed into the loop β is equivalent to the assertion that α is homotopic to β relative to $\{0,1\}$. One can actually prove that in general, homotopy behaves well with respect to compositions of maps and that the relation of homotopy on the set of all loops based at the same point is an equivalence relation; we refer to these equivalence classes as homotopy classes. Denote the homotopy class of a loop α by $< \alpha >$.

Theorem C.2. *(see e.g. [Ams83, Theorem 5.5]) The set of homotopy classes of loops in X based at v forms a group under the multiplication $< \alpha > \cdot < \beta > = < \alpha \cdot \beta >$. This is the* fundamental group $\pi_1(X, v)$ *of X based at v.*

Usually we write $\pi_1(X)$ for the fundamental group of X, ignoring the base point v. It turns out that the definition of the fundamental group of X actually depends only on its 2-skeleton X^2 and so $\pi_1(X) = \pi_1(X^2)$.

Let $X = X^2$ be a CW-complex equipped with an order $\prec_i$ among its i-cells. Let N_i be the number of i-cells of X. Sometimes it might be convenient to denote vertex (0-cell) v by $\{v, v\}$. Denote an edge (1-cell) joining two vertices v_i, v_j by $\{v_i, v_j\}$ (w.l.o.g. any two verices are joined by at most one edge). The 2-frontier path ∂C_i (of a 2-cell C_i) can be expressed in the form $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}, v_{i_1}\}$, if $\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots, \{v_{i_k}, v_{i_1}\}$ is the sequence of edges forming the k-polygon ∂C_i with $v_{i_1} \prec_0 v_{i_2} \prec_0 v_{i_m}$ for $3 \leq m \leq k$.

Definition C.3. *An* edge path *in a CW-complex X is a sequence $w = \{v_1, v_2, \dots, v_k\}$ of vertices in which each concecutive pair $\{v_i, v_{i+1}\}$ is either an edge of X, or $v_i = v_{i+1}$.*

Equivalently we can write the edge path w as $w = \{v_1, v_2\}\{v_2, v_3\} \cdots \{v_{k-1}, v_k\}$. If $v_i = v_k = v$, we then have an *edge loop* based at v. It is possible to include paths having no edges, i.e, paths consisting of an isolating vertex as an edge path.

Definition C.4. *Two closed edge paths (loops) w_1, w_2 based at $v_1 = v$ are* combinatorial homotopic *(or deformable) if they can be transformed from one to each other by finitely many elementary combinatorial deformations of the following type:*

(A) Insert or remove an edge which runs back and forth;

(B) Insert or remove the frontier path of a 2-cell C_i;

(C) Allow to change a repeated vertex $\{\dots, v, v, \dots\}$ to $\{\dots, v, \dots\}$ and vice versa;

(D) In an edge path w we can replace

$$\{v_{i_1}, v_{i_2}, \dots, v_{i_m}\} \longleftrightarrow \{v_{i_1}, v_{i_k}, \dots, v_{i_{m+1}}, v_{i_m}\}, \qquad 2 \leq m \leq k,$$

if $\{v_{i_1}, v_{i_2}, \dots, v_{i_m}, \dots, v_{i_{k-1}}, v_{i_k}, v_{i_1}\}$ is a frontier path of a 2-cell C_i, in other words we can let the path jump over the bounding polygon.

Observe that we can obtain (D) by first applying (B) and then (A) repeatedly. The endpoint(s) of an edge path always remain *fixed* under combinatorial deformation.

Deformation (C) allow us to eliminate point paths between edges, for example, $\{v_i, v_j, v_j, v_k\} = \{v_i, v_j, v_k\}$, or equivalently $\{v_i, v_j\}\{v_j, v_j\}\{v_j, v_k\} = \{v_i, v_j\}\{v_j, v_k\}$.

Consider, for instance, the case when $\{v_{i_1}, v_{i_2}\}\{v_{i_2}, v_{i_3}\} \cdots \{v_{i_4}, v_{i_5}\}\{v_{i_5}, v_{i_1}\}$ is a 2-frontier path; then by (D) we can replace

$$\{v_{i_1}, v_{i_2}\}\{v_{i_2}, v_{i_3}\} \longleftrightarrow \{v_{i_1}, v_{i_5}\}\{v_{i_5}, v_{i_4}\}\{v_{i_4}, v_{i_3}\}.$$

Define the *equivalence class* (or combinatorially homotopic class) of the edge path $w = \{v_0, v_{i_1}, \dots, v_0\}$ by $[w] = [v_0, v_{i_1}, \dots, v_0]$.

Definition C.5. *The* Edge Path Group $E_p(X, v) = E_p$ *of a* CW*-complex* X*, is the set of combinatorially homotopic classes of closed edge paths based at* v*, under usual path multiplication:*

$$[v_0, v_{i_1}, \dots, v_{i_k}, v_0][v_0, v_{j_1}, \dots, v_{i_r}, v_0] = [v_0, v_{i_1}, \dots, v_{i_k}, v_0, v_{j_1}, \dots, v_{i_r}, v_0].$$

In the specific case when the the CW-complex X is actually a simplicial complex (i.e, the frontier paths of all 2-cells are triangles) we write $E_p(X, v) = E_s(X, v)$. Using the simplicial approximation theorem (see e.g [Ams83, Theorem 6.7]), one can prove that $E_s(X, v) = \pi_1(X, v)$ (see e.g [Ams83, Theorem 6.10]).

This result can be generalized for arbitrary CW-complexes.

Theorem C.6. *[ST80, Mas77, Hat02]* $E_p(X, v) = \pi_1(X, v)$*, where* X *is an arbitrary (not necessarily simpicial)* CW*-complex.*

Scetch of Proof (for the full details see [ST80, §45].) One way to prove this theorem is as follows.

(I) Reduce general case of arbitrary X to the case of a simplicial complex K by repeatedly subdividing the CW-complex and showing that the edge path group E_p does not change during each subdivision. Distinguish two types of subdivision in order to obtain a CW-complex X' from X:

Subdivision of an edge: an inner point of an edge of the CW-complex X is made into a new vertex, decomposing the edge into two subedges.

Subdivision of closed 2-cell: consider the 2-cell C_i and its characteristic map $\Phi_i : D_i^2 \longrightarrow X$, such that $\Phi_i(D_i^2) = cl(C_i)$. The disc D_i^2 is decomposed by a chord into two parts. The endpoints of the chords transform under Φ_i restricted to $bd(D_i^2)$ to vertices of X'. The chord transform to a new edge and the two parts of the disc tranform to two new closed 2-cells $cl(C_{i_1})$ and $cl(C_{i_2})$ of X', such that their union gives $cl(C_i)$.

(II) Use the Edge Path Group theorem for simplicial complexes. □

Suppose that we have chosen a maximal tree $Y \subset X$. In other words Y is a 1-dimensional subcomplex of X, containing all the vertices of X, which is path connected and simply connected. Hence, edge loops in Y will not contribute to $E_p(X,v)$ and so we can effectively ignore the edges of Y from our calculations.

Definition C.7. *Write $G(X,Y)$ for the group which is determined by generators $g_{i,j}$, one for each directed edge $\{v_i,v_j\}$ subject to the relations:*

(1) $g_{i,j}=1$ *if* $\{v_i,v_j\}\in Y$;

(2) $g_{i_1,i_2}g_{i_2,i_3}\cdots g_{i_{m-1},i_m}g_{i_m,i_1}=1$ $\big($*OR* $g_{i_1,i_2}\cdots g_{i_{t-1},i_m}=g_{i_1,i_m}\cdots g_{i_{t+1},i_t}, 2\le t\le m-1\big)$ *if* $\{v_{i_1},v_{i_2},\dots,v_{i_m},v_{i_1}\}$ *is a 2-frontier path.*

Note that in case $i=j$ we get $\{v_i,v_j\}=v_i\in Y$ and so by type (1) relation above we get $g_{i,i}=1$. Since the edge path $\{v_i,v_j\}\{v_j,v_i\}$ is equivalent to the point path $\{v_i,v_i\}=v_i$ we get the *trivial relations*: $g_{i,j}\,g_{j,i}=g_{i,i}=1$ and $g_{j,i}\,g_{i,j}=g_{j,j}=1$.

Remark C.8. *Actually we only need to introduce one generator $g_{i,j}$ for each (undirected) edge $\{v_i,v_j\}$ with $i<j$ and use the trivial relations $g_{i,j}\,g_{j,i}=g_{j,i}\,g_{i,j}=1$ to get*

$$g_{j,i}=g_{i,j}^{-1}.$$

Observe that the number of generators is equal to N_1, the number of 1-cells of the CW-complex X, and the number of relations of type (2) is equal to N_2, the number of 2-cells of X.

Theorem C.9. *(see, e.g [ST80, Ams83, Hat02]* $G(X,Y)$ *is isomorphic to* $E_p(X,v)$.

Proof. We shall construct homomorphisms

$$\phi: G(X,Y)\longrightarrow E_p(X,v),\qquad \theta: E_p(X,v)\longrightarrow G(X,Y)$$

which are inverse to one another. First join base vertex $v=v_0$ to each vertex v_i of X by an edge path $E_i\subset Y$, taking $E_0=v$. Then define ϕ on the generators of $G(X,Y)$ by:

$$\phi(g_{i,j})=[E_iv_iv_jE_j^{-1}];$$

it follows that if $R_i=g_{i_1,i_2}\cdots g_{i_k,i_{k+1}}$ then $\phi(R_i)=\prod_{r=1}^{k}\phi(g_{i_r,i_{r+1}})$.

If $\{v_i,v_j\}\subset Y$ then $[E_iv_iv_jE_j^{-1}]$ is an edge loop which lies entirely in Y and therefore represents the identity element of $E_p(X,v)$ since Y is simply connected (this of course is also true when $i=j$).

Also, if $\{v_{i_1},v_{i_2},\dots,v_{i_m},v_{i_1}\}$ forms a frontier path of a 2-cell of X, then

$$\phi(g_{i_1,i_2})\,\phi(g_{i_2,i_3})\cdots\phi(g_{i_{m-1},i_m})\,\phi(g_{i_m,i_1})=$$

$$=[E_{i_1}v_{i_1}v_{i_2}E_{i_2}^{-1}]\,[E_{i_2}v_{i_2}v_{i_3}E_{i_3}^{-1}]\cdots[E_{i_{m-1}}v_{i_{m-1}}v_{i_m}E_{i_m}^{-1}]\,[E_{i_m}v_{i_m}v_{i_1}E_{i_1}^{-1}]\;=$$

$$= [E_{i_1} v_{i_1} v_{i_2} E_{i_2}^{-1} E_{i_2} v_{i_2} v_{i_3} E_{i_3}^{-1} \cdots E_{i_{m-1}} v_{i_{m-1}} v_{i_m} E_{i_m}^{-1} E_{i_m} v_{i_m} v_{i_1} E_{i_1}^{-1}] =$$

$$= [E_{i_1} \underbrace{v_{i_1} v_{i_2} v_{i_3} \cdots v_{i_{m-1}} v_{i_m} v_{i_1}}_{\text{2-frontier path: remove by } (B)} E_{i_1}^{-1}] = [E_{i_1} E_{i_1}^{-1}] = 1$$

$$= \phi(g_{i_1,i_2}, g_{i_2,i_3}, \ldots, g_{i_m,i_1}).$$

So the relations in $G(X,Y)$ are preserved and ϕ defines a homomorphism from $G(X,Y)$ to $E_p(X,v)$.

We now define $\theta : E_p(X,v) \longrightarrow G(X,Y)$ by

$$\theta([v_{i_0} v_{i_1} \cdots v_{i_m} v_{i_0}]) = g_{i_0,i_1} g_{i_1,i_2} \cdots g_{i_m,i_0}$$

and we show that is indeed a homomorphism:

$$\theta([v_0, v_{k_1}, v_{l_1}, \ldots, v_{n_1}, v_0])\theta([v_0, v_{k_2}, v_{l_2}, \ldots, v_{n_2}, v_0]) =$$

$$= (g_{0,k_1} g_{k_1,l_1} \cdots g_{n_1,0})(g_{0,k_2} g_{k_2,l_2} \cdots g_{n_2,0}) =$$

$$= (g_{0,k_1} g_{k_1,l_1} \cdots g_{n_1,0} g_{0,k_2} g_{k_2,l_2} \cdots g_{n_2,0}) =$$

$$= \theta([v_0, v_{k_1}, v_{l_1}, \ldots, v_{n_1}, v_0, v_{k_2}, v_{l_2}, \ldots, v_{n_2}, v_0]).$$

Observe that

$$\theta\,\phi(g_{i,j}) = \theta([E_i v_i v_j E_j^{-1}]) = g_{i,j},$$

since all the pairs of vertices in E_i and E_i^{-1} belong to Y. So, $\theta\,\phi$ is the identity on $G(X,Y)$. Moreover, for any edge loop $[v_0, v_k, v_l, \ldots, v_n, v_0]$ we have:

$$[v_0, v_k, v_l, \ldots, v_n, v_0] = [E_0 v_0 v_k E_k^{-1}]\,[E_k v_k v_l E_l^{-1}] \cdots [E_n v_n v_0 E_0^{-1}],$$

and

$$\phi\,\theta([E_m v_m v_s E_s^{-1}]) = \phi(\underbrace{\cdots}_{1} g_{m,s} \underbrace{\cdots}_{1}) = [E_m v_m v_s E_s^{-1}].$$

Thus, $\phi\theta$ is the identity on each of the terms in this product and therefore $\phi\,\theta$ is the identity homomorphism on $E_p(X,v)$. □

Relations of type (1) result from all generators corresponding to edges in Y (= a maximal tree of X^1) becoming trivial. There are $N_0 - 1$ such edges. Hence, the number of non-trivial generators for the fundamental group $\pi_1(X)$ is

$$N_1 - (N_0 - 1) = 1 + N_1 - N_0.$$

We conclude with a summary of the above method for computing the fundamental group of a CW-complex.

Summary: Suppose X is a finite CW-complex and Y a maximal tree of the 1-skeleton X^1. Choose vertex $v \in Y$. For each edge $\{v_i, v_j\} \in X^1 \setminus Y$ choose a loop $A_{i,j} = \{E_i v_i v_j E_j^{-1}\}$ where E_k is a unique path in Y, joining v to v_k . Then $\pi_1(X^1, v)$ is generated by homotopy classes of $A_{i,j}$'s, say $[A_{i,j}] = g_m$, $1 \le m \le t = 1 + N_1 - N_0$. Let $\phi_1, \dots, \phi_{N_2} : S^1 \longrightarrow X^1$ be attaching maps of 2-cells of X. We can assume w.l.o.g. that $\phi_i(1)$ is a vertex v_j, least with respect to $\prec_0$ among all vertices of the polygon that bounds $C_i \approx \phi_i\big(int(D^2)\big)$. For each ϕ_i express the homotopy class of the loops $E_j\, \phi_i\, E_j^{-1}$ as a product of powers of generators g_m. Note that the attaching maps ϕ_i are circular loops in X^1; the loop $E_j\, \phi_i\, E_j^{-1}$ may not be nullhomotopic in X^1 but certainly is in X^2 after the 2-cell C_i is attached. Let R_i $(1 \le i \le N_2)$ be the words in letters $g_1, \dots, g_t$ obtained in this way. Then

$$\pi_1(X) = \big\langle\, g_1, \dots, g_t \,|\, R_1, \dots, R_{N_2} \,\big\rangle.$$

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